GROUP ACTIONS WITH TNI-CENTRALIZERS

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Throughout this presentation all groups are finite.

Let G be a group acted on by the group A.

Question

How does the nature of the action of A
(e.g. the way $C_G(A) = \{g \in G : g^a = g \ for \ all \ a \in A\}$
is embedded in G)
influence the structure of G?

Fixed point free action

(Thompson, 1959) A group admitting a fpf automorphism of prime order is nilpotent.

(Rowley, 1995) If A acts coprimely and fixed point freely on G then G is solvable.

(Belyaev-Hartley, 1996) If a nilpotent group A acts fixed point freely on G then G is solvable.

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Length Type Problems

Problems of finding some bounds for the invariants of a solvable group, like the derived length, p-length, nilpotent (Fitting) length by using the given information about the group.

(started by Hall-Higman in 1956)

A Typical Answer (Turull, 1984)

Let A and G be both solvable. If A acts coprimely on G then

$$f(G) \leq f(C_G(A)) + 2\ell(A)$$

and this bound is the best possible.

Here $f(G)$ stands for the nilpotent (Fitting) length of G and $\ell(A)$ is the number of primes, counted with multiplicities, dividing $|A|$.

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Longstanding conjectures when A is fixed point free: Coprime case

(Thompson) Let A act on G fixed point freely. If $|A|$ is a prime, then G is solvable with $f(G) = 1 = \ell(A)$.

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Longstanding conjectures when A is fixed point free: Noncoprime case

(Bell-Hartley, 1990) Given nonnilpotent A and a positive integer k. Then there is a solvable G such that A acts fixed point freely and noncoprimely on G, and $f(G) \geq k$.

(Hartley-Belyaev, 1996) If A is nilpotent and acts fixed point freely on G, then G is solvable.

Conjecture II

Let A be a nilpotent group acting fixed point freely on G. Then $f(G) \leq \ell(A)$.

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Turull, Kei-Nah, Espuelas and others
made contributions to the study on these conjectures.

Turull settled Conjecture 1 for almost all A in a sequence of papers.

Conjecture I is TRUE when

A acts with regular orbits, that is, there exists $v \in S$ such that $C_A(v) = C_A(S)$ for each elementary abelian A-invariant section S of G.

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Fixed Point Free Action: Noncoprime Case

Theorem

(Dade, 1969)
Let a nilpotent group A act fixed point freely on the group G. Then $f(G) \leq 10(2^{\ell(A)} - 1) - 4\ell(A)$.

In the same paper, Dade conjectured that $f(G) \leq c\ell(A)$ for some constant c.

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When A is a Frobenius Group with fixed point free kernel

A question of Mazurov initiated the study of the case where $A = FH$ is a Frobenius group with kernel F and complement H, and $C_G(F) = 1$.

The dependence of certain invariants such as the order, the rank, the nilpotent length, the nilpotency class and the exponent of the group G on the corresponding invariants of $C_G(H)$ have been studied by Khukhro, Makarenko and Shumyatsky.

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Theorem

Suppose that a group G admits a Frobenius group FH of automorphims with kernel F and complement H such that $C_G(F) = 1$. Then

(i) $F_k(C_G(H)) = F_k(G) \cap C_G(H)$ for all k, and

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Definition

Let F be a nontrivial nilpotent group acted on by a nontrivial group H via automorphisms so that the condition $[F, h] = F$ holds for all nonidentity elements $h \in H$.

We call the semidirect product FH a Frobenius-like group.

Remark

The group FH is Frobenius-like if and only if F is a nontrivial nilpotent group and the group FH/F' is Frobenius with kernel F/F' and complement isomorphic to H.

A Generalization - Frobenius-like Groups

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(2014) We proved

Theorem

Let G be a group admitting a Frobenius-like group of automorphisms FH of odd order such that F' is of prime order and $[F', H] = 1$. Assume that $C_G(F) = 1$ and $(|G|, |H|) = 1$. Then

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Let FH be a Frobenius-like group with complement H of prime order p where $C_F(H)$ is of prime order . Suppose that FH acts on a p'-group G by automorphims such that $C_G(F) = 1$. Then

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TNI-subgroups

Definition

Let H be a subgroup of a group G.

We call H a trivial normalizer intersection subgroup of G if $N_G(H) \cap H^g = 1$ for any $g \in G \setminus N_G(H)$.

- Every normal subgroup is TNI.
- $TNI \Rightarrow TI$.
- Every Hall subgroup which is TI is also TNI.

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- *Let H be a TNI-subgroup of G. Then for any subgroup K of G, $K \cap H$ is a TNI -subgroup of K.*

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Coprime action of a group with TNI-centralizer

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Theorem

Let A be a group that acts coprimely on the group G.
If $C_G(A)$ is a solvable TNI-subgroup of G, then
G is solvable.

Remark

Coprimeness is necessary:
Let $G = A_5$ and τ_σ be the inner automorphism of G induced by $\sigma = (1, 2, 3, 4, 5)$.
Now, $C_G(\tau_\sigma) = \langle \sigma \rangle$ is solvable and TNI, but G is not solvable.

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Theorem

Let A be a coprime automorphism of prime order of a solvable group G such that $C_G(A)$ is a TNI-subgroup of G.
Then
$f(G) \leq f(C_G(A)) + 1$. In particular, $f(G) \leq 4$ when $C_G(A)$ is nonnormal.

:) This can be regarded as a generalization of the Thompson's result which asserts that a group admitting a fpf automorphism of prime order is nilpotent.

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A-tower

Let $f(G) = n$. As $(|G|, |A|) = 1$, there exists a sequence $\hat{P}_1, \dots, \hat{P}_n$ of subgroups of G where

(a) $\hat{P}_i$ is an A-invariant p_i-subgroup, p_i is a prime, $p_i \neq p_{i+1}$,

(b) $\hat{P}_i \leq N_G(\hat{P}_j)$ whenever $i \leq j$;

(c) $P_n = \hat{P}_n$ and $P_i = \hat{P}_i / C_{\hat{P}_i}(P_{i+1})$ for $i = 1, \dots, n-1$ and $P_i \neq 1$ for $i = 1, \dots, n$;

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Passing to a tower-like sequence inside $C_G(A)$

Theorem

(Turull,1984) Let A be a group of prime order acting on a group G with $(|A|, |G|) = 1$. Let $\hat{P}_i,\ i = 1, \ldots, n$, be an A-tower and assume that A centralizes $\hat{P}_k$, (possibly with $k = 0$ and $\hat{P}_k = 1$). Then there exists $j \geq k$ such that the sequence $(C_{\hat{P}_i}(A))$, $i = 1, \ldots, j-1, j+1, \ldots, n$ satisfies conditions $(a), (b), (c)$ of the definition of A-tower, except the condition that $p_i \neq p_{i+1}$. If 2 does not divide $|P_1|$ we may take $j > k$.

Proof

Let $f(G) = n$. Assume first that $C_G(A)$ is normal in G.

The fixed point free action of A on the group $G/C_G(A)$ yields that $G/C_G(A)$ is nilpotent by the well known theorem of Thompson. Then $f(G) \leq f(C_G(A)) + 1$ which is not the case. Therefore we may assume that $C_G(A)$ is nonnormal in G and hence there exists a section S/T of G on which the action of $C_G(A)$ is Frobenius.

There is an A-tower $\hat{P}_i, i = 1, \ldots, n$ of subgroups of G since the action is coprime. By induction we have $G = \prod_{i=1}^{n} \hat{P}_i$.
By the Theorem above there exists i such that the sequence

$$C_{\hat{P}_n}(A), \ldots, C_{\hat{P}_{i+1}}(A),\ C_{\hat{P}_{i-1}}(A), \ldots, C_{\hat{P}_1}(A)$$

is a tower with the exception that p_{i-1} may be equal to p_{i+1}.

Proof

If $P = C_{\hat{P}_n}(A) \neq 1$, then $S/T = [S,P]T/T \leq \hat{P}_nT/T \cap S/T = 1$ which is impossible. It follows that $P = 1$ and hence $i = n$, that is $f(G) \leq f(C_G(A)) + 1$.

Finally suppose that $C_G(A)$ is nonnormal. Then $C_G(A)$ is a Frobenius complement, and hence $f(C_G(A)) \leq 3$. It follows that $f(G) \leq 4$ as desired. $\square$

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Action of a Frobenius Group with fixed point free kernel

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(2017) We also obtained

Theorem

Let G be a solvable group on which a Frobenius group FH, with kernel F and complement H, acts coprimely. If $C_G(F)$ is a TNI-subgroup of G, then

$f([G,F]) = f(C_{[G,F]}(H))$.

In particular $f(G) \leq f(C_G(H)) + f(C_G(F))$.

THANK YOU