Tame blocks

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How we look at blocks

- When we say: "B is a block" we mean: "B is a block of RG for **some group** G".
- What can be said about **all** blocks that share a given defect group D?

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- $p = 2$, D (semi-)dihedral or quaternion: **tame representation type** ($\rightsquigarrow$ we call B a **tame block**).
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- Brauer & Olsson studied the character theory of tame blocks (number of characters, their height, etc.).
- Erdmann classified all "algebras of (semi-)dihedral or quaternion type". This class of algebras
  - **is defined in representation theoretic terms.** *Defining properties: symmetric, indecomposable, tame rep. type, non-singular Cartan matrix and conditions on the shape of its "stable Auslander-Reiten components".*
  - **contains all tame blocks**, but also algebras which aren't blocks.

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...and the notions of equivalence that come with them

Let A and B be two R-algebras (conditions as above). We call A and B

Morita equivalent		**derived equivalent**		**stably equivalent**
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And an equivalence just for blocks

Two blocks A and B with a common defect group D can also be **Puig equivalent**, which is even stronger than merely being Morita equivalent.

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- Usually, one takes R to be a field. But these conjectures also make sense (and are stronger) over a p-adic ring, assuming R is "big enough".
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- None of this is completely settled for tame blocks!

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Question

Let R be a p-adic ring, $\pi \in R$ a prime element. Assume $R/\pi R$ is algebraically closed. Given two bocks A and B, is the following true?

$$A/\pi A \sim_{\mathrm{Morita}} B/\pi B \overset{?}{\Longrightarrow} A \sim_{\mathrm{Morita}} B \qquad (\star)$$

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Remark

- Part of the Q_8-case was already settled by Holm-Kessar-Linckelmann.
- The only blocks of quaternion defect where $(\star)$ is open are those with two simple modules. That is exactly where Donovan conjecture is as yet unsolved.

Lifting to a p-adic ring (a rough sketch of how $(\star)$ is proved)

Let R be a p-adic ring, $\pi \in R$ a prime and $k = R/\pi R$.

- When certain invariants of a block B (the "decomposition numbers") are $\leqslant 1$, then there is a good chance of describing all R-algebras A (having some properties known to hold for block algebras) with $A/\pi A \sim_{\mathrm{Morita}} B/\pi B$. (Plesken)

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- Given two k-algebras $C \sim_{\mathrm{derived}} D$, then there is a bijection between

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- Conclusion: The property $(\star)$, $A/\pi A \sim_{\mathrm{Morita}} B/\pi B$ implies $A \sim_{\mathrm{Morita}} B$, is essentially a property of the derived equivalence class of $A/\pi A$. We just need to prove it for one representative to which the "decomposition numbers $\leqslant 1$"-methods apply and show uniqueness there.

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Remark

This method also allowed to show that certain algebras in Erdmann's classification do **not** occur as blocks. Now we know exactly which algebras occur as blocks with dihedral defect groups.

Self-equivalences

For a block B we have a chain of groups of self-equivalences

$$\{\text{Morita self-eq.}\} \longrightarrow \{\text{“standard” derived self-eq.}\} \longrightarrow \{\text{stable self-eq. (of “Morita type”)}\}$$

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Problem

What do the above groups of self-equivalences look like?

Remark: τ-tilting theory (Adachi-Iyama-Reiten)

The problem reduces to describing all "tilting complexes" over tame blocks. Among these "tilting complexes", those of length two are parametrised by a class of modules (in the case of a block, or, a symmetric algebra):

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Derived self-equivalences for tame blocks

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Conclusion

In the situation of tame blocks, this is enough to give generators for the groups of ("standard") derived self-equivalences (Aihara-Mizuno).

Question

Can we describe the group of stable self-equivalences of "Morita type" of tame blocks?

Thank you for your attention!