Coprime automorphisms acting with nilpotent centralizers

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Fixed-point subgroup

Definition

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- Similarly, if $A \leq Aut(G)$ we denote by $C_G(A)$ the subgroup

$$C_G(A) = \{x \in G; x^a = x \text{ for all } a \in A\}.$$

  If $C_G(A) = 1$, we say that A is a *fixed-point-free* group of automorphisms of G.

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- If A is a non-cyclic abelian group of automorphisms of a finite group G and $(|A|,|G|)=1$, then $G=\langle C_G(a)\ ;\ a\in A^{\#}\rangle$.

Well-known results...

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- (*A. Turull-1984*) If a finite group G admits an automorphism of prime order such that $C_G(\varphi)$ is nilpotent, then its Fitting height is at most 3.
- (*A. Turull-1984*) Let G be a finite soluble group admitting a soluble group of automorphisms A such that $(|G|, |A|) = 1$. Then $h(G) \leq h(C_G(A)) + 2k(A)$, where k denotes the number of primes whose products gives $|A|$.

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- (*P. Shumyatsky-2001*) Let p be a prime, A an elementary abelian p-group of order p^3 and G a finite p'-group. Suppose that A acts on G in such a way that $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. Then G is nilpotent with class bounded in terms of c and p.

Non-abelian case

Theorem (Shumyatsky, de Melo - 2016)

Let p be a prime and A a finite group of exponent p acting on a finite p'-group G. Assume that A has order at least p^3 and $C_G(a)$ is nilpotent of class at most c for any nontrivial element of A. Then G is nilpotent with class bounded in terms of c and p.

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- We may assume that $A = (C_p \times C_p) \rtimes C_p$. That is, A is an extra-special group of order p^3 and $|Z(A)| = p$.

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- G_0 is A-invariant and $\overline{A} = A/Z(A)$ is an elementary abelian group of order p^2. Thus, $G_0 = \langle C_G(\overline{a}) \ ; \ \overline{a} \in \overline{A}^{\#} \rangle$.

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- On the other hand, for any subgroup$B$ of order p^2 of A we have $G = \langle C_G(b) \; ; \; b \in B \rangle$. Then $[G, G_0] = 1$.

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- Therefore, $G/Z(G)$ admits a fixed-point-free automorphism of order p.

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- Consider the associated Lie ring of the group G

$$L(G) = \bigoplus_{i=1}^{n} \gamma_i/\gamma_{i+1},$$

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- Then we can prove that there exists a (c, p)-bounded number u such that $[L, \underbrace{L_0, \ldots, L_0}_{u}] = 0$.

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- Thus, the Lie ring $L(G)$ is soluble with (c, p)-bounded derived length.
- Now, we can assume that $L(G)$ is metabelian and then we prove that $L(G)$ is nilpotent with (c, p)-bounded class.

Thank you!