Refining Brauer's Induction Theorem

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Joint work with Anton Evseev.

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Global-Local Conjectures are one of the key research areas in modern Representation Theory. These all involve the general notion that the Representation Theory of a finite group G is controlled/described by the Representation Theory of its local subgroups, i.e. its Sylow p-subgroups and their normalizers in the group. For example:

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Conjecture (McKay Conjecture)

Let G be a finite group and $P \in \mathrm{Syl}_p(G)$ for a prime p. We denote by $\mathrm{Irr}_{p'}(G)$ to be the set of irreducible characters of G of order not divisible by p. Then,

$$\left|\mathrm{Irr}_{p'}(G)\right| = \left|\mathrm{Irr}_{p'}(N_G(P))\right|.$$

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$$M_l(G) := \left|\{\chi \in \mathrm{Irr}(G) : \chi(1) \equiv \pm l \mod p\}\right|.$$

Then, $M_l(G) = M_l(N_G(P))$ for every l coprime to p.

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While producing his own refinement of this conjecture, Evseev introduced the following 3 properties, which may or may not hold for a pair (G, H).

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- $\mathcal{C}^p(G) := \mathbb{Z}\left[\mathrm{Irr}^p(G)\right]$.
- $\mathcal{I}\left(G, P, \mathcal{S}\right)$ is the $\mathbb{Z}$-span of induced characters from subgroups in $\mathcal{S}$ which is essentially a set of subgroups of P.

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WIRC-Syl There exists a signed bijection $F : \pm\mathrm{Irr}_{p'}(G) \to \pm\mathrm{Irr}_{p'}(H)$, such that

$$F(\chi) \equiv \mathrm{Res}^G_H \chi \mod \mathcal{C}^p(H) + \mathcal{I}(H, P, \mathcal{S}),$$

for all $\chi \in \pm\mathrm{Irr}_{p'}(G)$.

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Note that the Isaacs-Navarro refinement of the McKay Conjecture follows if WIRC-Syl holds for $(G, N_G(P))$.

Brauer's Induction Theorem

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Theorem

Let G be a finite group and suppose that $\chi \in \mathrm{Irr}(G)$. We denote by $\mathrm{El}(G)$, the set of elementary subgroups of G. Then there exists $\theta_{(E)} \in \mathcal{C}(E)$ for each $E \in \mathrm{El}(G)$ such that,

$$\chi = \sum_{E \in \mathrm{El}(G)} \mathrm{Ind}_E^G \theta_{(E)}.$$

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Conjecture

Suppose that G is a finite group and $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \equiv 0 \mod p^a$. Then $\chi \in \mathcal{A}_{p^a}(G, Z(G))$.

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- Späth reduction of the Alperin-McKay Conjecture (2013).

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(3) Can we shift the Representation Theory of G to that of N?
This can be approached in several different ways. Our approach was to combine an induction hypothesis with Clifford Theory to introduce a character θ of N, whose Representation Theory could be intimately linked to our character of G. In our situation, we then use the character theory of direct products to introduce characters of S.

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(6) If $a = 1$ and $p = 2$ and the abelianisation of the Sylow 2-subgroup is elementary abelian, then the conjecture is true.