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A generalization of the Hall-Witt identity

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An introduction

The source of our inspiration for this research was the following paragraph from Isaacs' book:

"An amazing commutator formula is the Hall-Witt identity:

$$[[x_3, x_1^{-1}], x_2]^{x_1}[[x_1, x_2^{-1}], x_3]^{x_2}[[x_2, x_3^{-1}], x_1]^{x_3} = 1,$$

which holds for any three elements of every group. . . One can think of the Hall-Witt formula as a kind of three-variable version of the much more elementary two-variable identity, $[x_1, x_2][x_2, x_1] = 1$. This observation hints at the possibility that a corresponding four-variable formula might exist, but if there is such a four-variable identity, it has yet to be discovered."

— Martin Isaacs, Finite Group Theory, page 125

If we wish to generalize the Hall-Witt identity, the natural question is:

What do we mean by a generalization of the Hall-Witt identity?

We derived our approach to this problem from the structure of the Hall-Witt identity:

$$[[x_3, x_1^{-1}], x_2]^{x_1}[[x_1, x_2^{-1}], x_3]^{x_2}[[x_2, x_3^{-1}], x_1]^{x_3} = 1.$$

This identity consists of three factors, the first one, or the ***generator*** of the identity, being $[[x_3, x_1^{-1}], x_2]^{x_1}$ and the next two factors are obtained by applying the permutations $\sigma = (1\ 2\ 3)$ and σ^2 to the indices of the generator, respectively. Indeed, we have

$$[[x_3, x_1{}^{-1}], x_2]^{x_1} \overset{\sigma = (1\ 2\ 3)}{\curvearrowright} [[x_{\sigma(3)}, x_{\sigma(1)}{}^{-1}], x_{\sigma(2)}]^{x_{\sigma(1)}} = [[x_1, x_2{}^{-1}], x_3]^{x_2}$$

$$\overset{\sigma = (1\ 2\ 3)}{\curvearrowright} [[x_{\sigma(1)}, x_{\sigma(2)}{}^{-1}], x_{\sigma(3)}]^{x_{\sigma(2)}} = [[x_2, x_3{}^{-1}], x_1]^{x_3}$$

Thus, denoting the generator of the Hall-Witt identity by W, we may write the identity in the following form:

$$\mathrm{W}(\sigma \cdot \mathrm{W})(\sigma^2 \cdot \mathrm{W}) = 1$$

where

$$\sigma \cdot \mathrm{W} = [[x_{\sigma(3)}, x_{\sigma(1)}^{-1}], x_{\sigma(2)}]^{x_{\sigma(1)}}$$

and

$$\sigma^2 \cdot \mathrm{W} = [[x_{\sigma^2(3)}, x_{\sigma^2(1)}^{-1}], x_{\sigma^2(2)}]^{x_{\sigma^2(1)}}$$

In the spirit of this example, we defined a **broad generalization** of the Hall-Witt identity, using the language of Free Groups.

Preliminaries: Free Group

Let $\Sigma_n = \{x_1, x_2, \ldots, x_n\}$ be a finite set of symbols, called an ***alphabet***. By a ***word*** over the alphabet Σ_n we mean any finite sequence of elements, which can be written from the symbols of $\Sigma_n \cup \Sigma_n^{-1}$, including the empty word 1, where $\Sigma_n^{-1} = \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$.

We shall denote by Σ_n^* the ***free group*** generated by Σ_n. This group consists of *all* words which can be built over the alphabet Σ_n. The operation in this group is the ***concatenation***, i.e. appending the second factor to the first one. Two words are different, unless their equality follows from the group axioms. For example,

$$xx^{-1} = x^{-1}x = 1 \quad \text{but} \quad x_1x_2 = x_1x_3x_3^{-1}x_2 \neq x_2x_1.$$

The Hall-Witt property

The "broad generalization" of the Hall-Witt identity are words $\mathrm{W} \in \Sigma_n^*$ which satisfy the identity

$$(*) \quad \mathrm{W}_\sigma := \mathrm{W}(\sigma \cdot \mathrm{W})(\sigma^2 \cdot \mathrm{W}) \cdots (\sigma^{k-1} \cdot \mathrm{W}) = 1,$$

where $\sigma \in \mathrm{S}_n$ of *order* k and σ^j act on W by permuting the indices of the elements in W.

Such words W will be called words which satisfy the ***Hall-Witt property*** with respect to σ. The word W in $(*)$ will be called ***the generator*** of the identity $\mathrm{W}_\sigma = 1$. In particular, we have seen that the word $\mathrm{W} = [[x_3, x_1^{-1}], x_2]^{x_1}$ satisfies the Hall-Witt property with respect to $\sigma = (1\ 2\ 3)$.

Our aim in this paper is three-fold.

(a) To find words W in Σ_n^* and corresponding $\sigma \in \mathrm{S}_n$, such that W will have the Hall-Witt property with respect to σ *for an infinite number* of values of n.

Example 1 Let

$$\mathrm{W} = x_1 x_2 \cdots x_n x_1^{-1} x_n^{-1} x_{n-1}^{-1} \cdots x_2^{-1}$$

with respect to $\sigma = (1\ 2\ \ldots\ n)$. This W satisfies $\mathrm{W}_\sigma = 1$ for $n \geqslant 2$, as we shall show later.

(b) To find words $\mathrm{A} \in \Sigma_n^*$, so that the ***special words*** $\mathrm{W} = [\mathrm{A}, x_1]$ and $\mathrm{W} = \mathrm{A}^{x_1}$, reminiscent of the generator of the original Hall-Witt identity, and the corresponding $\sigma \in \mathrm{S}_n$, will have the Hall-Witt property ***for an infinite number*** of values of n.

Example 2 Let

$$\mathrm{A} = x_2 x_3 \cdots x_n$$

and

$$\mathrm{W} = [\mathrm{A}, x_1] = [x_2 x_3 \cdots x_n, x_1]$$

with respect to $\sigma = (1\ 2\ \ldots\ n)$. This W satisfies $\mathrm{W}_\sigma = 1$ for $n \geqslant 2$, as we shall show later.

Example 3 Let

$$\mathrm{W} = [x_1^{-1}, x_2 x_3 \ldots x_n]^{x_1}$$

with respect to $\sigma = (1\ 2\ \ldots\ n)$. This W satisfies $\mathrm{W}_\sigma = 1$ for $n \geqslant 2$, as we shall show later.

If $n = 2$, we get

$$[x_1^{-1}, x_2]^{x_1} [x_2^{-1}, x_1]^{x_2} = 1.$$

We consider this identity as a candidate for the **two variable** Hall-Witt identity.

Our third aim in this paper, and the most important one, is:

(c) To find special words W in Σ_n^*, as described in (b), and corresponding $\sigma = (1\ 2\ \ldots\ n)$ in S_n, such that W will have the Hall-Witt property with respect to σ **for an infinite number** of values of n, and the original Hall-Witt generator $[[x_3, x_1^{-1}], x_2]^{x_1}$ appears as W for certain value of n.

Such words will be called ***generators of a generalized Hall-Witt identity***.

Example 4 Let

$$\mathrm{W} = [x_4^{x_3} x_5^{x_4} x_6^{x_5} \cdots x_n^{x_{n-1}} [x_n, x_1^{-1}], x_2]^{x_1}$$

with respect to $\sigma = (1\ 2\ \ldots\ n)$.
If we apply the convention that if $n = 3$, then the empty product $x_4^{x_3} x_5^{x_4} x_6^{x_5} \cdots x_n^{x_{n-1}}$ equals 1, then W satisfies $\mathrm{W}_\sigma = 1$ for $n \geqslant 3$, as we shall show later.
The first three members of the family are

$$\mathrm{W} = [[x_3, x_1^{-1}], x_2]^{x_1} \quad \text{for } n = 3,$$
$$\mathrm{W} = [x_4^{x_3} [x_4, x_1^{-1}], x_2]^{x_1} \quad \text{for } n = 4,$$
$$\mathrm{W} = [x_4^{x_3} x_5^{x_4} [x_5, x_1^{-1}], x_2]^{x_1} \quad \text{for } n = 5.$$

The identities corresponding to the first two members are:

$$[[x_3, x_1^{-1}], x_2]^{x_1}[[x_1, x_2^{-1}], x_3]^{x_2}[[x_2, x_3^{-1}], x_1]^{x_3} = 1$$

which is the original Hall-Witt identity, and

$$[x_4^{x_3}[x_4, x_1^{-1}], x_2]^{x_1}[x_1^{x_4}[x_1, x_2^{-1}], x_3]^{x_2}[x_2^{x_1}[x_2, x_3^{-1}], x_4]^{x_3}[x_3^{x_2}[x_3, x_4^{-1}], x_1]^{x_4} = 1$$

which may be viewed as a "four-variable analog" of the Hall-Witt identity. Thus this $\mathrm{W} \in \Sigma_n^*$ is a generator of a generalized Hall-Witt identity.

Under what conditions $\mathrm{W} \in \Sigma^*$ satisfies the Hall-Witt property?

We have proved the following theorem:

Theorem 1

Let $\mathrm{W} \in \Sigma^*$ *and* $\sigma \in \mathrm{S}_n$. *Then* $\mathrm{W}_\sigma = 1$ *if and only if there exist a word* $\mathrm{U} \in \Sigma^*$ *such that* $\mathrm{W} = \mathrm{U}(\sigma \cdot \mathrm{U}^{-1})$.

Theorem 1 tells us how to construct words which satisfy the Hall-Witt property.

We continue with some examples mentioned above.

Example 1 (continued) Let

$$\mathrm{W} = x_1 x_2 \cdots x_n x_1^{-1} x_n^{-1} x_{n-1}^{-1} \cdots x_2^{-1}$$

with respect to $\sigma = (1\ 2\ \ldots\ n)$.

Since

$$\mathrm{W} = x_1 x_2 \cdots x_n (\sigma \cdot (x_1 x_2 \cdots x_n)^{-1}),$$

it follows from Theorem 1 that W satisfies $\mathrm{W}_\sigma = 1$ for $n \geqslant 2$, as claimed.

Example 2 (continued) Let

$$\mathrm{W} = [x_2 x_3 \cdots x_n, x_1]$$

with respect to $\sigma = (1\ 2\ \ldots\ n)$. Since

$$\begin{aligned} \mathrm{W} &= (x_2 x_3 \cdots x_n)^{-1} x_1^{-1} (x_2 x_3 \cdots x_n) x_1 \\ &= (x_n^{-1} \ldots x_2^{-1} x_1^{-1})(x_2 x_3 \ldots x_n x_1) \\ &= \mathrm{U}(\sigma \cdot \mathrm{U}^{-1}) \end{aligned}$$

for $\mathrm{U} = x_n^{-1} \ldots x_2^{-1} x_1^{-1}$, it follows by Theorem 1 that W satisfies $\mathrm{W}_\sigma = 1$ for $n \geqslant 2$, as claimed. Hence

$$[x_2 x_3 \cdots x_n, x_1][x_3 x_4 \cdots x_n x_1, x_2] \cdots [x_1 x_2 \cdots x_{n-1}, x_n] = 1.$$

Theorem 1 can be also used in order to verify whether a given word satisfies the Hall-Witt property. As an example, let us consider the word $\mathrm{W} = [[x_3, x_1^{-1}], x_2]^{x_1}$ from the original Hall-Witt identity. In this case

$$\mathrm{W} = [[x_3, x_1^{-1}], x_2]^{x_1} = x_3^{-1} x_1^{-1} x_3 x_2^{-1} x_3^{-1} x_1 x_3 x_1^{-1} x_2 x_1.$$

As one can see W is of the form $\mathrm{U}(\sigma \cdot \mathrm{U}^{-1})$, where $\mathrm{U} = x_3^{-1} x_1^{-1} x_3 x_2^{-1} x_3^{-1}$ and $\sigma = (1\ 2\ 3)$. So indeed, W satisfies the Hall-Witt property.

However, notice that Theorem 1 does not tell us how **to find** the special Hall-Witt identities. The rest of this lecture will be dedicated to this problem.

Special Hall-Witt Type Identities: Generator of the form $\mathrm{W} = [\mathrm{A}, x]$

In Example 2 we saw a Hall-Witt type identity with a generator of the form

$$\mathrm{W} = [x_2 x_3 \cdots x_n, x_1].$$

This motivated us to find **all** Hall-Witt type identities with generators of the form

$$\mathrm{W} = [\mathrm{A}, x]$$

where $x \in \Sigma_n = \{x_1, x_2, \ldots, x_n\}$ and $\mathrm{A} \in \Sigma_n^*$.

The following theorem completely determines the appropriate A.

Theorem 2

Let $\Sigma = \{x_1, x_2, \ldots, x_n\}$ *and let* $\mathrm{A} \in \Sigma^*$ *and* $i_0 \in \{1, 2, \ldots, n\}$. *Set* $\mathrm{W} = [\mathrm{A}, x_{i_0}]$ *and* $\sigma \in \mathrm{S}_n$. *Then* $\mathrm{W}_\sigma = 1$ *if and only if either* $\mathrm{A} = x_{i_0}^m$, *where* $m \in \mathbb{Z}$ *and* σ *is an arbitrary permutation, or*

$$\mathrm{A} = x_{i_0}^m (x_{i_0} x_{i_1} x_{i_2} \cdots x_{i_\ell})^k \quad \text{and} \quad (i_0\ i_1\ i_2\ \ldots\ i_\ell) \in \mathrm{dcd}(\sigma),$$

where m *and* k *are integers,* $k \neq 0$, $1 \leqslant \ell \leqslant n - 1$ *and* $i_0, i_1, \ldots, i_\ell \in \{1, 2, \ldots, n\}$ *are distinct indices.*

Comment: In this context, $\tau \in \mathrm{dcd}(\sigma)$ will mean that τ is one of the cycles in the *disjoint cycle decomposition* of σ.

Example 2 (continued) By taking $\sigma = (1\ 2\ 3\ \ldots\ n)$, $l = n - 1$, $m = -1$, $i_0 = 1$ and $k = 1$ we obtain the Hall-Witt type identity with the generator

$$\mathrm{W} = [x_1^{-1}(x_1 x_2 \ldots x_n), x_1] = [x_2 x_3 \ldots x_n, x_1],$$

as defined in Example 2.

Example 5 By taking $\sigma = (1\ 2\ 3\ \ldots\ n)$, $\ell = n-1$, $m = 0$ we obtain the Hall-Witt type identity with the generator

$$\mathrm{W} = [(x_1 x_2 \cdots x_n)^k, x_1]$$

In particular, over the alphabet $\Sigma = \{x_1, x_2, x_3, x_4\}$ and for *any* non-zero integer k, we obtain the generator

$$\mathrm{W} = [(x_1 x_2 x_3 x_4)^k, x_1]$$

which yields the following Hall-Witt type identity

$$[(x_1x_2x_3x_4)^k, x_1][(x_2x_3x_4x_1)^k, x_2][(x_3x_4x_1x_2)^k, x_3][(x_4x_1x_2x_3)^k, x_4] = 1$$

Special Hall-Witt Type Identities: Generator of the form $\mathrm{W} = \mathrm{A}^x$

Since the generator of the original Hall-Witt identity is

$$\mathrm{W} = [[x_3, x_1^{-1}]x_2]^{x_1},$$

in order to obtain an analog of the original Hall-Witt identity it is reasonable to search for Hall-Witt identities with generators of the form

$$\mathrm{W} = \mathrm{A}^x,$$

where $x \in \Sigma_n = \{x_1, x_2, \ldots, x_n\}$ and $\mathrm{A} \in \Sigma_n^*$.

The following theorem completely determines the appropriate A.

Theorem 3

Let $\Sigma = \{x_1, x_2, \dots, x_n\}$ *and let* $i \in \{1, 2, \dots, n\}$, $\mathrm{A} \in \Sigma^*$ *and let* $\sigma \in \mathrm{S}_n$. *Set* $\mathrm{W} = \mathrm{A}^{x_i}$. *Then* $\mathrm{W}_\sigma = 1$ *if and only if one of the following conditions is satisfied*

(a) σ *is arbitrary and* $\mathrm{A} = 1$.

(b) $\sigma(i) = i$ *and there is a word* $\mathrm{L} \in \Sigma^*$ *such that* $\mathrm{A} = \mathrm{L}(\sigma \cdot \mathrm{L}^{-1})$.

(c) $\sigma(i) \neq i$ *and there is a word* $\mathrm{L} \in \Sigma^*$ *such that*

$\mathrm{A} = x_i x_j^{-1} \mathrm{L}(\sigma \cdot \mathrm{L}^{-1})$ *where* $j \in \{1, 2, \dots, n\}$ *satisfies* $i = \sigma(j)$ *or*

$\mathrm{A} = (\sigma^{-1} \cdot \mathrm{L})\mathrm{L}^{-1} x_j x_i^{-1}$ *where* $j \in \{1, 2, \dots, n\}$ *satisfies* $\sigma(i) = j$ *or*

$\mathrm{A} = x_i \mathrm{L}(\sigma \cdot \mathrm{L}^{-1}) x_i^{-1}$.

Example 6 Over the alphabet $\Sigma = \{x_1, x_2, x_3\}$ let us choose $i = 1$,

$$\mathrm{L} = x_1^{-1} x_3 x_2^{-1} x_3^{-1} \quad \text{and} \quad \sigma = (1\ 2\ 3).$$

Here $\sigma(3) = 1$, so by part (c) of Theorem 3 (first option) we obtain

$$\begin{aligned}
\mathrm{A} &= x_1 x_3^{-1} \mathrm{L}(\sigma \cdot \mathrm{L}^{-1}) \\
&= x_1 x_3^{-1}(x_1^{-1} x_3 x_2^{-1} x_3^{-1})((1\ 2\ 3) \cdot x_3 x_2 x_3^{-1} x_1) \\
&= x_1 x_3^{-1}(x_1^{-1} x_3 x_2^{-1} x_3^{-1})(x_1 x_3 x_1^{-1} x_2) \\
&= (x_1 x_3^{-1} x_1^{-1} x_3) x_2^{-1}(x_3^{-1} x_1 x_3 x_1^{-1}) x_2 \\
&= (x_3^{-1} x_1 x_3 x_1^{-1})^{-1} x_2^{-1}(x_3^{-1} x_1 x_3 x_1^{-1}) x_2 \\
&= [x_3^{-1} x_1 x_3 x_1^{-1}, x_2] \\
&= [[x_3, x_1^{-1}], x_2]
\end{aligned}$$

Thus

$$\mathrm{W} = \mathrm{A}^{x_1} = [[x_3, x_1^{-1}], x_2]^{x_1}$$

which yields the original Hall-Witt identity.

Example 4 (continued) Let $n \geqslant 4$ be an integer and consider the alphabet $\Sigma = \{x_1, x_2, \dots, x_n\}$. If $\sigma = (1\ 2\ 3\ \dots\ n)$, $i = 1$ and

$$\mathrm{L} = x_1^{-1}(x_n x_{n-1}^{-1} x_n^{-1}) \cdots (x_4 x_3^{-1} x_4^{-1})(x_3 x_2^{-1} x_3^{-1})$$

then it can be shown that

$$\begin{aligned} \mathrm{A} &= x_1 x_n^{-1} \mathrm{L}(\sigma \cdot \mathrm{L}^{-1}) \\ &= [x_4^{x_3} x_5^{x_4} \cdots x_n^{x_{n-1}} [x_n, x_1^{-1}], x_2] \end{aligned}$$

is of the type required by Theorem 3.
Thus

$$\mathrm{W} = \mathrm{A}^{x_1} = \left[x_4^{x_3} x_5^{x_4} x_6^{x_5} \cdots x_n^{x_{n-1}} [x_n, x_1^{-1}]\,,\ x_2\right]^{x_1}$$

satisfies the Hall-Witt property. Notice that if we apply the convention that the empty product equals 1, then we may use this expression also with $n = 3$ to obtain

$$\mathrm{W} = [[x_3, x_1^{-1}], x_2]^{x_1},$$

which generates the original Hall-Witt identity. Therefore W is a generator of a generalized Hall-Witt identity.

Example 3 (continued) Let $n \geqslant 2$ be an integer and consider the alphabet $\Sigma = \{x_1, x_2, \ldots, x_n\}$. If $\sigma = (1\ 2\ 3\ \ldots\ n)$, $i = 1$ and

$$\mathrm{L} = x_{n-1}^{-1} \ldots x_2^{-1} x_1^{-1}$$

then it can be shown that

$$\begin{aligned} \mathrm{A} &= x_1 x_n^{-1} \mathrm{L}(\sigma \cdot \mathrm{L}^{-1}) \\ &= \left[x_1^{-1}, x_2 x_3 \ldots x_n\right] \end{aligned}$$

is of the type required by theorem 3.
Thus

$$\mathrm{W} = \mathrm{A}^{x_1} = \left[x_1^{-1}, x_2 x_3 \ldots x_n\right]^{x_1}$$

satisfies the Hall-Witt property. Note that assigning $n = 2$ yields the identity

$$[x_1^{-1}, x_2]^{x_1} [x_2^{-1}, x_1]^{x_2} = 1$$

which may be viewed as a "two-variable analog" of the Hall-Witt identity.

Example 7 Let $n \geqslant 1$ be an integer and consider the alphabet $\Sigma = \{x_1, x_2, \ldots, x_n\}$. If $\sigma = (1\ 2\ 3\ \ldots\ 2n+1)$, $i = 1$ and

$$\mathrm{L} = x_{2n-1}^{-1} \cdots x_5^{-1} x_3^{-1} x_1^{-1} x_{2n+1} x_{2n}^{-1} \cdots x_6^{-1} x_4^{-1} x_2^{-1} x_{2n+1}^{-1}$$

then it can be shown that

$$\begin{aligned} \mathrm{A} &= x_1 x_{2n+1}^{-1} \mathrm{L}(\sigma \cdot \mathrm{L}^{-1}) \\ &= \left[(x_1 x_3 x_5 \cdots x_{2n-1})^{x_{2n+1}} x_1^{-1}, x_2 x_4 x_6 \cdots x_{2n}\right] \end{aligned}$$

is of the type required by Theorem 3. Thus

$$\mathrm{W} = \mathrm{A}^{x_1} = \left[(x_1 x_3 x_5 \cdots x_{2n-1})^{x_{2n+1}} x_1^{-1}\,,\ x_2 x_4 x_6 \cdots x_{2n}\right]^{x_1}$$

satisfies the Hall-Witt property. Note that if we take $n = 1$ we get the generator of the original Hall-Witt identity:

$$\mathrm{W} = [x_1^{x_3} x_1^{-1}\,,\ x_2]^{x_1} = [[x_3, x_1^{-1}], x_2]^{x_1}$$

Thus also W is a generator of a generalized Hall-Witt identity.

This is the END of my talk.

I would be happy to receive your comments.
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THANK YOU for your ATTENTION!