On algebraic normalizers of maximal tori in simple groups of Lie type

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Introduction

Consider a simple algebraic group $\overline{G}$ of adjoint type over an algebraic closed field $\overline{F}_p$ of positive characteristic p.
Suppose that σ is a Stainberg map (i.e. $|\overline{G}_\sigma| < \infty$). Denote $O^{p'}(\overline{G}_\sigma)'$ by G, it is called a finite group of Lie type.
Let $\overline{T}$ be a maximal σ-invariant torus of $\overline{G}$, then $T := \overline{T} \cap G$ is a maximal torus of G and $N(G, T) := N_{\overline{G}}(\overline{T}) \cap G$ is the algebraic normalizer of T in G.
In this talk we specify, when $N(G, T)$ is equal to $N_G(T)$.

Motivation and preliminary results

In "Seminar on Algebraic Groups and Related Finite Groups" T. A. Springer and R. Stainberg consider reductive algebraic groups and show that if no root relative to $\overline{T}$ vanishes on T, then $N(G, T)$ is equal to $N_G(T)$. Then they notice that it would be worthwhile to work out the exact exceptions.

Lemma 1, T. A. Springer and R. Stainberg

If $\overline{T}$ is σ-invariant, $W = \overline{N}/\overline{T}$ is its Weyl group, then the classes of maximal tori fixed by σ under conjugation by G are in one-one correspondence with the classes of σ-conjugate elements of W.

Elements w_1 and w_2 are called σ-conjugate if $w_1 = (w^{-1})^\sigma w_2 w$ for some $w \in W$.

Motivation and preliminary results

Let π be the natural homomorphism from $\overline{N}$ no W.
Fix a maximal σ-invariant torus $\overline{T}$, then $\overline{T}^g$ is σ-invariant if and only if $g^\sigma g^{-1} \in \overline{N}$. The map $\overline{T}^g \mapsto \pi(g^\sigma g^{-1})$ is a bijection between G-conjugacy classes of maximal σ-invariant tori and classes of σ-conjugate elements of W.

Lemma 2

Let $g^\sigma g^{-1} \in \overline{N}$ and $\pi(g^\sigma g^{-1}) = w$. Then $(\overline{T}^g)_\sigma = (\overline{T}_{\sigma w})^g$.

Classical groups, $G = PSL_n^{\varepsilon}(q)$

Let $G = PSL_n^{\varepsilon}(q)$, then W is isomorphic to S_n and σ acts trivial on it. Denote by $(n_1)(n_2)\dots(n_k)$; $n_1 \leqslant n_2 \leqslant \dots \leqslant n_k$; $\sum_{i=1}^{k} n_i = n$ the class of elements which are the product of k independent cycles of orders n_i.
For example $(1,\dots,n_1)(n_1+1,\dots,n_1+n_2)\dots(n-n_k+1,\dots,n)$.

Lemma 3 (Buturlakin and Grechkoseeva, 2007)

Let $\overline{G} = SL_n(\overline{F}_q)$, $\overline{T}$ be a group of all diagonal matrices with determinant 1 and w be the standard representative of type $(n_1)(n_2)\dots(n_k)$, U be the subgroup of $GL_n(\overline{F}_q)$ of all block-diagonal matrices $bd(D_1,\dots,D_k)$ such that $D_i = diag(\lambda_i, \lambda_i^{\varepsilon q}, \dots, \lambda_i^{(\varepsilon q)^{n_i-1}})$, $\lambda_i^{(\varepsilon q)^{n_i}-1} = 1$, for all $i \in \{1,2,\dots,k\}$. Then $\overline{T}_{\sigma w} = U \cap \overline{G}$.

Results for $PSL_n^{\varepsilon}(q)$

$N(G,T) = N_G(T)$ except cases when

$G = PSL_n^{+}(2)$; w is of type $\overbrace{(1)(1)\dots(1)}^{\geqslant 2}(\dots)$

Classical groups, Simplectic groups

Let $G = PSp_{2n}(q)$, then W is isomorphic to Sl_n – the group of permutations τ on the set $\{1, 2, \ldots, n, -1, -2, \ldots, -n\}$ with $\tau(\imath) = -\tau(i)$ and σ acts trivial on it.
Omitting the signs, we obtain a homomorphism to S_n. Let $\tau \in Sl_n$ is mapped to cycle $(i_1 i_2 \ldots i_s)$, if $\tau^s(i_1) = i_1$ then τ is called a positive cycle of length s; if $\tau^s(i_1) = -i_1$ then τ is called a negative cycle of length s.
An element of Sl_n is uniquely determined by its decomposition on independent positive and negative cycles.
Denote by $(n_1)(n_2)\ldots(n_s)(\overline{n_{s+1}})(\overline{n_{s+2}})\ldots(\overline{n_k})$; $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$; $\sum_{i=1}^{k} n_i = n$ the class of elements which are the product of k independent cycles.

Classical groups, $G = PSp_n(q)$

Lemma 4, (Buturlakin and Grechkoseeva, 2007)

Let $\overline{G} = Sp_{2n}(\overline{F}_q)$, $\overline{T}$ be the group of all nonsingular matrices $bd(D, D^{-1})$, where D is diagonal matrix, and w be the standard representative of type $(n_1)(n_2)\dots(n_s)(\overline{n_{s+1}})(\overline{n_{s+2}})\dots(\overline{n_k})$. Let U be the subgroup of $Sp_{2n}(\overline{F}_q)$ of all block-diagonal matrices $bd(D_1, \dots, D_k, D_1^{-1}, \dots, D_k^{-1})$ such that $D_i = diag(\lambda_i, \lambda_i^q, \dots, \lambda_i^{(q)^{n_i-1}})$, $\lambda_i^{(q)^{n_i} - \varepsilon_i 1} = 1$, for all $i \in \{1, 2, \dots, k\}$. Then $\overline{T}_{\sigma w} = U$.

Results for $PSp_{2n}(q)$

$N(G, T) = N_G(T)$ except cases when
$G = PSp_{2n}(2)$; w is of type $(1)(\dots)$
$G = PSp_{2n}(3)$; w is of type $(1)(\dots)$

Classical groups, orthogonal groups of characteristic

If $G = \Omega_{2n+1}(q)$ then W is isomorphic to Sl_n and σ acts trivially on it.
If $G = P\Omega^{\varepsilon}_{2n}(q)$ then W is isomorphic to the subgroup of Sl_n consisting of permutations with even number of odd cycles in the decomposition on independent cycles.
If $\varepsilon = +$ then σ acts trivially on W.
Let $n_0 \in SO_{2n}(\overline{F}_q)$ be permutation matrix corresponding to negative cycle $(n, -n)$ and denote $\pi(n_0) \in W$ by w_0.
If $\varepsilon = -$ then w_1 and w_2 are σ-conjugate in W if and only if $w_0 w_1$ and $w_0 w_2$ conjugate in W.

Results for orthogonal groups of odd characteristic

$N(G, T) = N_G(T)$ with no exceptions.

Classical groups, orthogonal groups of characteristic

Results for orthogonal groups of even characteristic

$N(G,T) = N_G(T)$ except cases when

$G = P\Omega^{+}_{2n}(2)$; w is of type $\overbrace{(1)(1)\dots(1)}^{\geqslant 2}(\dots)$

$G = P\Omega^{-}_{2n}(2)$; $w_0 w$ is of type $\overbrace{(1)(1)\dots(1)}^{\geqslant 2}(\dots)$

$\Omega_{2n+1}(q)$ is isomorphic to $PSp_{2n}(q)$, so the exceptions are the same:

$G = \Omega_{2n+1}(2)$; w is of type $(1)(\dots)$

Exceptional groups

We know that $N(G,T) = N_G(T)$ if and only if there are no root subgroups in $C_{\overline{G}}(T)^0$. Assume that there is a root subgroup in $\overline{R} = C_{\overline{G}}(T)^0$. Then $\overline{R}$ is a reductive subgroup of maximal rank and $\overline{R} \cap G$ is a reductive subgroup of maximal rank in G. It is clear that $T \leqslant Z(R)$, so $|T|$ divides $|Z(R)|$.

[1] *D. I. Deriziotis*, Conjugacy classes and centralizers of semisimple elements in finite groups of Lie type, Fachbereich Mathematik, Universität Essen, 1984

[2] *D. I. Deriziotis*, The Centralizers of Semisimple Elements of the Chevalley Groups E_7 and E_8, TOKYO J. MATH. VOL. 6, NO. 1, 1983

Exceptional groups

Results for Exceptional groups

$N(G, T) = N_G(T)$ for all tori of groups
${}^2G_2(3^{2n+1})$;
${}^2B_2(2^{2n+1})$;
${}^2F_4(2^{2n+1})$;
$G_2(q)$, $q > 3$;
$F_4(q)$, $q > 3$;
${}^2E_6(q^2)$, $q > 3$;
$E_6(q)$, $q > 3$;
$E_7(q)$, $q > 3$;
$E_8(q)$, $q > 3$;