Finiteness conditions for the non-abelian tensor product of groups[1]

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Non-abelian tensor product of groups

Let G and H be groups each of which acts upon the other (on the right),

$$G \times H \to G,\ (g, h) \mapsto g^h;\ \ H \times G \to H,\ (h, g) \mapsto h^g$$

and on itself by conjugation, in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$g^{(h^{g_1})} = \left(\left(g^{g_1^{-1}}\right)^h\right)^{g_1} \quad \text{and} \quad h^{(g^{h_1})} = \left(\left(h^{h_1^{-1}}\right)^g\right)^{h_1}. \qquad (1)$$

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In this situation we say that G and H act *compatibly* on each other. Let H^φ be an extra copy of H, isomorphic via $\varphi : H \to H^\varphi,\ h \mapsto h^\varphi$, for all $h \in H$.

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In this situation we say that G and H act *compatibly* on each other. Let H^{φ} be an extra copy of H, isomorphic via $\varphi : H \to H^{\varphi},\ h \mapsto h^{\varphi}$, for all $h \in H$. Consider the group $\eta(G,H)$ defined in [Nak00] as

$$\eta(G,H) = \langle G, H^{\varphi} \mid \ [g,h^{\varphi}]^{g_1} = [g^{g_1},(h^{g_1})^{\varphi}],\ [g,h^{\varphi}]^{h_1^{\varphi}} = [g^{h_1},(h^{h_1})^{\varphi}],$$
$$\forall g, g_1 \in G,\ h, h_1 \in H \rangle.$$

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It is a well known fact (see [Nak00, Proposition 2.2]) that the subgroup $[G, H^{\varphi}]$ of $\eta(G, H)$ is canonically isomorphic with the *non-abelian tensor product* $G \otimes H$, as defined by R. Brown and J.-L. Loday in their seminal paper [BL87], the isomorphism being induced by $g \otimes h \mapsto [g, h^{\varphi}]$ (see also [EL95]).

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$$\eta(G, H) = ([G, H^{\varphi}] \cdot G) \cdot H^{\varphi}, \tag{2}$$

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We observe that the defining relations of the tensor product can be viewed as abstractions of commutator relations (see also [Kap99]).

Non-abelian tensor square of groups

We observe that when $G = H$ and all actions are conjugations, $\eta(G, H)$ becomes the group $\nu(G)$ introduced in [Roc91]. More precisely,

$$\nu(G) := \langle G, G^{\varphi} \mid [g_1, {g_2}^{\varphi}]^{g_3} = [{g_1}^{g_3}, ({g_2}^{g_3})^{\varphi}] = [g_1, {g_2}^{\varphi}]^{{g_3}^{\varphi}}, \ \ g_i \in G \rangle.$$

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In particular, $\nu(G) = ([G, G^{\varphi}] \cdot G) \cdot G^{\varphi}$, where $[G, G^{\varphi}]$ is isomorphic to $G \otimes G$, the non-abelian tensor square of G. In the notation of [NR94], we denote by $\Delta(G)$ the diagonal subgroup of the non-abelian tensor square $[G, G^{\varphi}]$, $\Delta(G) = \langle [g, g^{\varphi}] \mid \ g \in G \rangle$.

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$$\chi(G) := \langle G, G^{\varphi} \mid [g, g^{\varphi}] = 1, \ \forall g \in G \rangle.$$

Some Results

Let G and H be groups that act compatibly on each other.

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Now, consider $G = H$ and all actions are conjugations.

- (Parvizi and Niroomand, [PN12]) Suppose that G is a finitely generated group. If the non-abelian tensor square $[G, G^{\varphi}]$ is finite, then so is G.

Question

An element $\alpha \in \eta(G, H)$ is called a *tensor* if $\alpha = [a, b^{\varphi}]$ for suitable $a \in G$ and $b \in H$. If N and K are subgroups of G and H, respectively, let $T_{\otimes}(N, K)$ denote the set of all tensors $[a, b^{\varphi}]$ with $a \in N$ and $b \in K$. In particular, $[N, K^{\varphi}] = \langle T_{\otimes}(N, K) \rangle$.

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Question: *If we assume certain restrictions on the set $T_{\otimes}(G, H)$, how does this influence in the structure of the groups $[G, H^{\varphi}]$ or $\eta(G, H)$?*

Commutators and Tensors

In [Ros62] Rosenlicht proved that if N and K are subgroups of a group M, with N normal in M, and if the set of commutators $\{[n,k] \,:\, n \in N,\ k \in K\}$ is finite, then so is the commutator subgroup $[N,K]$. Under appropriate conditions we can extend this result to the subgroup $[N, K^{\varphi}]$ of $\eta(G,H)$.

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Theorem 1. Let G and H be groups that act compatibly on each other and suppose that N and K are subgroups of G and H, respectively, such that N is K-invariant and K is N-invariant. If the set $T_{\otimes}(N,K)$ is finite, then so is the subgroup $[N,K^{\varphi}]$ of $\eta(G,H)$. In particular, the set $T_{\otimes}(G,H)$ is finite if and only if $[G,H^{\varphi}]$ is finite.

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An immediate consequence of the above theorem is the finiteness criterion for the non-abelian tensor product of finite groups due to G. Ellis (see also [Tho10]).

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However, when $G = H$ and all actions are conjugations, we obtain the following result for the non-abelian tensor square:

Theorem 2. Let G be a group. The non-abelian tensor square $[G, G^{\varphi}]$ is finite if and only if G is a BFC-group and $[G^{ab}, (G^{ab})^{\varphi}]$ is finite.

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In the sequel we consider certain finiteness conditions for the group G in terms of the torsion elements of the non-abelian tensor square $[G, G^{\varphi}]$.

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Lemma. (Rocco, [Roc94]) Let G be a group with finitely generated abelianization. Suppose that the diagonal subgroup $\Delta(G)$ is periodic. Then the abelianization G^{ab} is finite.

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Theorem 3. Let G be a group with finitely generated abelianization.

(a) If the diagonal subgroup $\Delta(G)$ is periodic, then $\Delta(G)$ is finite. Moreover, the abelianization G^{ab} is isomorphic to a subgroup of the diagonal subgroup $\Delta(G)$.

(b) If π is a set of primes and the non-abelian tensor square $[G, G^{\varphi}]$ is a π-group, then so is the group G.

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Proposition 2. Let p be a prime and m a positive integer. Let G be a finitely generated locally graded group. Suppose that every tensor has order dividing p^m. Then G is finite.

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