On Some Generation Methods of Finite Simple Groups

Ayoub B. M. Basheer
Department of Mathematical Sciences, North-West University (Mafikeng), P Bag X2046, Mmabatho 2735, South Africa

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Abstract

In this talk we consider some methods of generating finite simple groups with the focus on *ranks of classes*, (p, q, r)*-generation* and *spread (exact)* of finite simple groups. We show some examples of results that were established by the author and his supervisor, Professor J. Moori on generations of some finite simple groups.

Introduction

- Generation of finite groups by suitable subsets is of great interest and has many applications to groups and their representations.
- For example, Di Martino and et al. [39] established a useful connection between generation of groups by conjugate elements and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of the sporadic simple groups.
- In view of applications, it is often important to exhibit generating pairs of some special kind, such as
  - generators carrying a geometric meaning,
  - generators of some prescribed order,
  - generators that offer an economical presentation of the group.
- The problem of investigation of generators for a finite group has a rich history, with numerous applications. The classification of finite simple groups is involved heavily and play a pivotal role in most general results on the generations of finite groups.

Introduction

- We recall that a finite group is said to be 2-*generated* if it can be generated by two suitable elements.
- It is well-known that finite non-abelian simple groups are 2-generated (Miller [60], Steinberg [69] and Aschbacher and Guralnick [7]). The latter showed that any sporadic simple group can be generated by an involution and another suitable element.
- A 2017 paper by C. King [56] gave a refinement where it was shown that every finite non-abelian simple group is generated by an involution and an element of a prime order.
- Turning to the maximal subgroups of finite simple groups, Burness et. al. [25] showed that any maximal subgroup of a non-abelian finite simple group is 4-generated or less and that this bound is best possible.
- The topic of generation of finite simple groups is fairly rich and we are trying to cover some of the classical and recent results. In the following we mention few kinds of generations of finite simple groups:

Introduction

- Generation by involutions satisfying certain conditions (J. Ward [73]).
- $\frac{3}{2}$-*generated* groups. Guralnick and Kantor [52] showed that every finite simple group is $\frac{3}{2}$-generated. Breuer et. al., [23] conjectured that any finite group is $\frac{3}{2}$-generated if and only if every proper quotient is cyclic. A recent work of Guralnick [50] reduces this conjecture to almost simple groups. A 2017 paper by S. Harper [54] extended the results to almost simple symplectic and odd-dimensional orthogonal groups.
- Computing the maximal size of an *irredundant generating set* of a finite group G, denoted by $m(G)$. In [74], Whiston found that $m(S_n) = n - 1$ and $m(A_n) \leq n - 2$, while in [24], Brooks found that $m(M_{11}) = 5$ and $m(M_{12}) = 6$.
- Guralnick et. al. [51, 52] were interested in probabilistic random generation of a finite simple group using elements of a fixed conjugacy class of the group. Burness, Guralnick, Kantor, Liebeck, Saxl and Shalev have a pioneering role in the problem of the probabilistic random generation.
- We are interested in generating finite simple groups via the *ranks*, (p, q, r)-*generations*, nX-*complementary generation* and *exact spread*.

Some Settings

- Let G be a finite group and $C_1, \cdots, C_k$ be $k \geq 3$ (not necessarily distinct) conjugacy classes of G with $g_1, \cdots, g_k$ being representatives for these classes respectively.
- For a fixed representative $g_k \in C_k$ and for $g_i \in C_i,\ 1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*.
- With $\mathrm{Irr}(G) = \{\chi_1, \cdots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula (see for example [58])

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}.$$

Some Settings

- Also for a fixed $g_k \in C_k$ we denote by $\Delta_G^*(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1} \rangle = G$.
- If $\Delta_G^*(C_1, \cdots, C_k) > 0$, the group G is said to be $(C_1, \cdots, C_k)$-*generated*.
- If $H \leq G$ is any subgroup containing a fixed element $g_k \in C_k$, we denote $\Sigma_H(C_1, C_2, \cdots, C_k)$ to be the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1} \rangle \leq H$.
- The value of $\Sigma_H(C_1, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, \cdots, c_k)$ of H-conjugacy classes $c_1, \cdots, c_k$ such that $c_i \subseteq H \bigcap C_i$.

Theorem (e.g. see Ganief [47])

Let G be a finite group and $H \leq G$ containing a fixed element x such that $\gcd(o(x), [N_G(H){:}H]) = 1$. Then the number $h(x, H)$ of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular

$$h(x, H) = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the G-class $[x]_G$.

- The number $h(x, H)$ is useful in giving a lower bound for $\Delta^*_G(C_1, \cdots, C_k)$, namely $\Delta^*_G(C_1, \cdots, C_k) \geq \Theta_G(C_1, \cdots, C_k)$, where

$$\Theta_G(C_1, \cdots, C_k) = \Delta_G(C_1, \cdots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, \cdots, C_k),$$

  g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, \cdots, C_k$.
- Except for $G = \mathbb{M}$ (the Monster group), $\Theta_G = \Theta_G(C_1, \cdots, C_k)$ can be computed for all the sporadic groups using GAP [48] or Magma [20].
- If $\Theta_G > 0$ then certainly G is $(C_1, \cdots, C_k)$-generated.

Some Results on Generation and Non-Generation

- Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \cdots, lX}_{m-times}, (nZ)^m)$-generated. If G is $(2X, mY, nZ)$-generated simple group, then G is $(mY, mY, (nZ)^2)$-generated.
- Let G be a finite centerless group. If $\Delta_G^*(C_1, \cdots, C_k) < |C_G(g_k)|,\ g_k \in C_k$, then $\Delta_G^*(C_1, \cdots, C_k) = 0$ and therefore G is not $(C_1, \cdots, C_k)$-generated.
- Ree and Scott Theorems for non-generation ([66] and [67]).

On the Ranks of Classes of Simple Groups

As mentioned, we are interested mainly in generating finite simple groups via the *ranks* of classes, (p, q, r)*-generation* and *spread* of a simple group.

Definition

Let G be a finite simple group and X be a non-trivial conjugacy class of G. The *rank* of X in G, denoted by $rank(G{:}X)$ is defined to be the minimal number of elements of X generating G.

- An application: Ranks of classes of finite groups are involved in computations of the covering number of the finite simple group (see Zisser [76]).
- In [61, 62, 64], J. Moori computed $rank(Fi_{22}{:}2X)$, for $X \in \{A, B, C\}$. He found that $rank(Fi_{22}{:}2B) = rank(Fi_{22}{:}2C) = 3$, while $rank(Fi_{22}{:}2A) \in \{5, 6\}$. The work of Hall and Soicher [53] implies that $rank(Fi_{22}{:}2A) = 6$. Then in a considerable number of publications (for example but not limited to, see [1, 2, 3, 4, 5, 6] or [61]) Moori, Ali and Ibrahim explored the ranks for various sporadic simple groups. In fact the determination of the ranks of the sporadic simple groups is almost completed.

On the Ranks of Some Classes of $A_n, \ n > 5$

We give some general results on the ranks for certain conjugacy classes of elements of the simple alternating group A_n. These results are due to the author with his supervisor in [15].

- The alternating group $A_n, \ n \geq 5$ has $\lfloor \frac{n}{3} \rfloor$ conjugacy classes of elements of order 3. The cycle structures of these classes are $3^m 1^{n-3m}, \ 1 \leq m \leq \lfloor \frac{n}{3} \rfloor$. For $m = 1$, let $3A$ denote the class of elements of A_n of cycle structure (a, b, c). We determine $rank(A_n{:}3A)$.
- Lemmas 3.1, 3.2 and 3.3 of [15] show that $rank(A_5{:}3A) = 2$, $rank(A_n{:}3A) \neq 2, \ \forall \ n \geq 6$ and $rank(A_6{:}3A) = 3$. These results have been used for the mathematical induction to show that:

Theorem

For the alternating group $A_n, \ n \geq 5$ we have

$$rank(A_n{:}3A) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

On the Ranks of Some Classes of A_n, $n > 5$

Also for the classes of n-cycles for n odd and $(n-1)$-cycles for n even of A_n, we established the following result:

Theorem

For $n \geq 5$, we have $\underbrace{rank(A_n{:}nX)}_{n \text{ is odd}} = 2 = \underbrace{rank(A_n{:}(n-1)X)}_{n \text{ is even}}, \; X \in \{A, B\}.$

Sketch of the proof

- Consider the case when n is odd and the even case follows easily.
- Consider class nA and treatment for class nB is similar. For $n \geq 5$ odd, let $\alpha = (1, 2, 3, \cdots, n) \in nA$ and let $\beta = (1, 4, 5, 6, 7, \cdots, n, 2, 3)$.
- Let $H := \langle \alpha, \beta \rangle$. Clearly $H \leq A_n$. We will show that $H \geq A_n$.
- We recall from Cameron [26] that if G is a primitive subgroup of S_n and contains a 3-cycle, then $G \geq A_n$. We aim to show that H is primitive in S_n and contains a 3-cycle element. For this we have

$$H \ni \alpha^{-1}\beta = (1, n, n-1, n-2, \cdots, 4, 3, 2)(1, 4, 5, 6, 7, \cdots, n, 2, 3) = (1, 3, n).$$

On the Ranks of Some Classes of $A_n,\ n > 5$

- Since H contains n-cycle elements, it follows by O'Nan-Scott Theorem (see for example Theorem 2.4 of Wilson [75]) that H can not be of type (i) or (ii) of maximal subgroups of S_n. Hence H is a primitive subgroup of S_n and since H contains a 3-cycle element, it follows that $H \geq A_n$. Thus $H = A_n$.
- The last step is to show that α and β are conjugate in H. This is easy since $\beta = \alpha^{(1,2,3)}$. ■

- The structure constant method together with the above results were used in [15] to determine the ranks of the classes of alternating groups A_8 and A_9. We concluded that for A_8, we have $rank(A_8{:}2A) = rank(A_8{:}2B) = rank(A_8{:}3A) = 4$ and $rank(A_8{:}nX) = 2,\ \forall\ nX \notin \{1A, 2A, 2B, 3A\}$, while for A_9, we have $rank(A_9{:}nA) = 4$ for $n \in \{2, 3\}$, $rank(A_9{:}2B) = 3$ and $rank(A_9{:}nX) = 2$ for all $nX \notin \{1A, 2A, 2B, 3A\}$.
- In [14] the author speculated that $rank(A_n{:}5A),\ \ n \geq 6$, is given by

$$rank(A_n{:}5A) = \begin{cases} \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n = 4k \text{ or } 4k+1, \\ \left\lceil \frac{n}{4} \right\rceil & \text{if } n = 4k+2 \text{ or } 4k+3. \end{cases} \quad (1)$$

On the Ranks of Some Classes of $A_n, \; n > 5$

- The author verified the correctness of this conjecture for $6 \leq n \leq 9$ theoretically and for up to $n \leq 50$ computationally using GAP [48]. Moreover, he speculated that

$$A_n = \begin{cases} \langle (1,2,3,4,5),(1,5,6,7,8),(1,9,10,11,12),\cdots, \\ (1,n-6,n-5,n-4,n-3),(1,n-3,n-2,n-1,n)\rangle & \text{if } n = 4k, \\ \langle (1,2,3,4,5),(1,6,7,8,9),(1,10,11,12,13),\cdots, \\ (1,n-7,n-6,n-5,n-4),(1,n-3,n-2,n-1,n)\rangle & \text{if } n = 4k+1, \\ \langle (1,2,3,4,5),(1,3,4,5,6),(1,7,8,9,10),\cdots, \\ (1,n-4,n-3,n-2,n-1),(1,n-3,n-2,n-1,n)\rangle & \text{if } n = 4k+2, \\ \langle (1,2,3,4,5),(1,4,5,6,7),(1,8,9,10,11),\cdots, \\ (1,n-5,n-4,n-3,n-2),(1,n-3,n-2,n-1,n)\rangle & \text{if } n = 4k+3. \end{cases} \tag{2}$$

On the (p,q,r)-Generation of Finite Simple Groups

Definition

For $l,m,n \in \mathbb{N} \setminus \{1\}$, a group G is said to be (l,m,n)*-generated* if G can be generated by two elements x and y with $o(x) = l, \ \ o(y) = m$ and $o(xy) = n$.

- In this case G is a quotient group of the triangular group $\Delta(l,m,n)$, where for $k_1, \cdots, k_n \in \mathbb{N} \setminus \{1\}$, the group $\Delta(k_1, \cdots, k_n)$ has the presentation:

$$\Delta(k_1, \cdots, k_n) = \left\langle x_1, \cdots, x_n | \ x_1^{k_1} = x_2^{k_2} = \cdots = x_n^{k_n} = x_1 x_2 \cdots x_n = 1 \right\rangle.$$

- It is well-known that $\Delta(l,m,n)$ is finite if and only if $1/l + 1/m + 1/n > 1$. (see [29] and [60]). Finite $\Delta(l,m,n)$ are:
  - $\Delta(1,n,n)$ the cyclic group of order n,
  - $\Delta(2,2,n) \cong D_{2n}$ the dihedral group of order $2n$,
  - $\Delta(2,3,3)$ the alternating group A_4,
  - $\Delta(2,3,4)$ the symmetric group S_4 and $\Delta(2,3,5)$ the alternating group A_5.

On the (p, q, r)- and nX-Complementary Generation of Finite Simple Groups

- The triangle groups have a remarkable wealth of interesting finite quotient groups (see [29, 68, 70]).
- We remark that a $(2, 3, 7)$-generated group G gives rise to compact Riemann surfaces of genus greater than 2 with automorphism groups of maximal order. Those $(2, 3, 7)$-generated group are called *Hurwitz groups* ([55] and [70]). A paper by Conder [27] gives an update on finite simple groups that are Hurwitz.

Definition

Let nX denote a general conjugacy class of G containing elements of order n. A group G is said to be *nX-complementary generated* if, given an arbitrary non-identity element $x \in G$, there exists a $y \in nX$ such that G is $\langle x, y \rangle$.

- Woldar [72] proved that every sporadic simple group G is pA-complementary generated, where p is the largest prime divisor of $|G|$.

On the (p, q, r)- and nX-Complementary Generation of Finite Simple Groups

- In [43, 44, 45, 46, 63, 65], Moori and Ganief established all possible (p, q, r)-generations and nX-complementary generations of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2 and Fi_{22}.
- The same was done for the sporadic groups Co_1, Th, $O^{'}N$, Ly, Suz and He in [8, 9, 10, 11, 12, 13, 31, 32, 33, 34, 35, 36, 37, 38] (Ashrafi, Darafsheh and Moghani).
- In [16] we established all the (p, q, r)-generations of the Matheiu group M_{22}.
- The structure constant method has been used extensively by all the above authors in establishing the (p, q, r)- and nX-complementary generations of finite simple groups.

On the Spread of Finite Simple Groups

Definition

For $r \in \mathbb{N}$, a finite non-abelian group G is said to have *spread* r, if for every set $\{x_1, x_2, \cdots, x_r\}$ of distinct non-trivial r elements of G, there exists an element y in G such that $\langle x_i, y \rangle = G$ for all i. We say that G has *exact spread* r, denoted by $s(G)$, if it has spread r but not $r + 1$.

- The concepts of spread and exact spread are of interest and have many applications to computational group theory (see for example [49]) and also when studying the generating graph of a group (see [57]).
- Following Woldar [71], the exact spread of the alternating groups is known for even degrees, while for the odd degrees the problem is still open and in this case $s(A_{2n+1})$ tends to infinity as n grows.

On the Spread of Finite Simple Groups

- Results on exact spread for non-abeian simple groups can be found for example in [22]. In particular for the sporadic groups, the exact spread is known in only two cases, namely $s(M_{11}) = 3$ (Woldar [71]) and $s(M_{23}) = 8064$ (Fairbairn [40]).
- There have been several improvement to the lower and upper bound of the exact spread by many different authors including Professor Moori (see for example [21, 22, 41]). They used probabilistic methods to establish reasonable lower and upper bounds for the exact spread $s(G)$ for each of the sporadic simple groups.

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