RATIONALITY OF GROUPS AND CENTERS OF INTEGRAL GROUP RINGS

Andreas Bächle

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VRIJE
UNIVERSITEIT
BRUSSEL

Notation.

G	*finite* group
$\mathbb{Z}G$	integral group ring of G
$\mathrm{U}(\mathbb{Z}G)$	group of units of $\mathbb{Z}G$

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x semi-rational in G $:\Leftrightarrow \exists\, m \in \mathbb{Z} \;\; \forall_{(j,o(x))=1} j \in \mathbb{Z} : \; x^j \sim x \;\text{ or }\; x^j \sim x^m$

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x *inverse semi-rational in* G $:\Leftrightarrow \forall_{\substack{j \in \mathbb{Z} \\ (j,o(x))=1}} : x^j \sim x$ or $x^j \sim x^{-1}$

G is called *rational* $:\Leftrightarrow \forall\, x \in G : x$ is rational in G
etc.

For $\chi \in \mathrm{Irr}(G)$, $x \in G$ set

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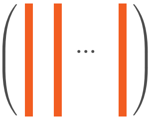
$\mathrm{CT}(G) =$ 

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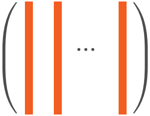
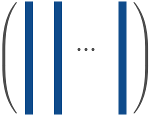
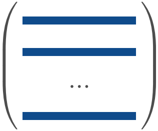
$\mathrm{CT}(G) =$ 

$\mathrm{CT}(G) =$ 

$\mathrm{CT}(G) =$ 

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Then $|\pi(S_n)| \longrightarrow \infty$ for $n \to \infty$.

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G solvable			

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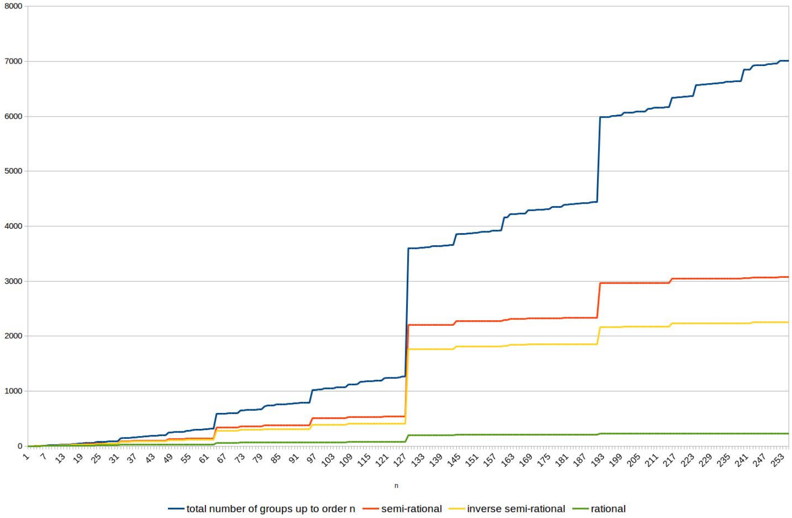

8000
7000
6000
5000
4000
3000
2000
1000
0
1 7 13 19 25 31 37 43 49 55 61 67 73 79 85 91 97 103 109 115 121 127 133 139 145 151 157 163 169 175 181 187 193 199 205 211 217 223 229 235 241 247 253
n
total number of groups up to order n
semi-rational
inverse semi-rational
rational

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$\mathrm{U}(\mathbb{Z}G)$

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$\nearrow$ often up to f.i. by "bicyclic units"

$\nwarrow$ covered by "bicyclic units" & "Bass units"

Theorem (Ritter-Sehgal, et.al.) For a finite group G TFAE

(1) G is cut.

(2) $\forall \chi \in \mathrm{Irr}(G)$: $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\sqrt{-d_\chi})$, $d_\chi \in \mathbb{Z}_{\geq 0}$.

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In particular: G cut $\Rightarrow$ G/N cut for all $N \trianglelefteq G$.

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(1) $2 \in \pi(G)$ or $3 \in \pi(G)$.

(2) If G is nilpotent, then G is a $\{2, 3\}$-group.

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Theorem (Maheshwary, 2016) Let G be a solvable cut group.

(1) If $|G|$ is odd $\Longrightarrow \pi(G) \subseteq \{3,7\}$ and
all elements of G are of prime power order.

(2) If $|G|$ is even and all elements of G are of prime power order
$\Longrightarrow \pi(G) \subseteq \{2,3,5,7\}$.

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Strategy of proof.

- $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$ (Chillag-Dolfi).
- Let G be a minimal counterexample, $V \trianglelefteq G$ minimal.
- Then $G \simeq V \rtimes G/V$, G/V is again cut.
- The $\mathbb{F}_{13}[G/V]$-module V has the "12-eigenvalue property".
- Derive restrictions on field of character values of V.
- By a result of Farias e Soares such a module cannot exist for a solvable group G/V. □

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Theorem (B., 2017). Let K be a Frobenius complement.

(1) If $|K|$ is even ...

(2) If $|K|$ is odd ...

Theorem (B., 2017). Let K be a Frobenius complement.

(1) If $|K|$ is even and the compelement of a cut Frobenius group G, then G is isomorphic to a group in the series on the left ($b, c, d \in \mathbb{Z}_{\geq 1}$) or one of the groups on the right.

(a) $C_3^b \rtimes C_2$	(α) $C_5^2 \rtimes Q_8$
(b) $C_3^{2b} \rtimes C_4$	(β) $C_5^2 \rtimes (C_3 \rtimes C_4)$
(c) $C_3^{2b} \rtimes Q_8$	(γ) $C_5^2 \rtimes \mathrm{SL}(2,3)$
(d) $C_5^c \rtimes C_4$	(δ) $C_7^2 \rtimes \mathrm{SL}(2,3)$
(e) $C_7^d \rtimes C_6$	
(f) $C_7^{2d} \rtimes (Q_8 \times C_3)$	

Conversely, for each of the above structure descriptions, there is a unique cut Frobenius group.

(2) If $|K|$ is odd ...

Theorem (B., 2017). Let K be a Frobenius complement.

(1) If $|K|$ is even ...

(2) If $|K|$ is odd, then there is a cut Frobenius group G if and only if $K \simeq C_3$ and the kernel F is a group admitting a fixed-point free automorphism σ of order 3 such that

(a) F is a cut 2-group.
In particular, $|F| = 2^{2a}$, $a \in \mathbb{Z}_{\geq 1}$ and F is an extension of an abelian group of exponent a divisor of 4 by an an abelian group of exponent a divisor of 4.

(b) F is an extension of an elementary abelian 7-group by an elementary abelian 7-group, $\exp F = 7$ and σ fixes each cyclic subgroup of F.

Strategy of proof. G cut Frobenius group with complement K.

- K is also cut.
- Show that K is solvable, so $\pi(G) \subseteq \{2, 3, 5, 7\}$.
- Determine possible structures of $P \in \mathrm{Syl}_p(K)$.
- Determine possible structures of K.
- Use irreducible representations of these complements to describe structure of some G.
- Decide which subdirect products of the groups above are cut Frobenius groups. □

References

A. Bächle, *Integral group rings of solvable groups with trivial central units*, 2017, arXiv:1701.04347[math.GR].

G.K. Bakshi, S. Maheshwary, I.B.S. Passi, *Integral group rings with all central units trivial*, J. Pure Appl. Algebra, **221**(8), 1955-1965, 2017, arXiv:1606.06860[math.RA].

D. Chillag, S. Dolfi, *Semi-rational solvable groups*, J. Group Theory **13**(4), 535-548, 2010.

S. Maheshwary, *Integral group rings with all central units trivial: solvable groups*, 2016, arXiv:1612.08344[math.RA].

J. Ritter, S.K. Sehgal, *Integral group rings with trivial central units*, Proc. Amer. Math. Soc. **108**(2), 327-329, 1990.