Generating pairs for the Fischer’s group $Fi_{23}$

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Abstract

A group $G$ is called $(l, m, n)$-generated, if it is a quotient of the triangle group $T(l, m, n) = \langle x, y, z \mid x^l = y^m = z^n = xyz = 1 \rangle$. Moor [13] posed the question of finding all the triples $(l, m, n)$ such that non-abelian finite simple groups are $(l, m, n)$-generated. In the present article, we answer this question for the Fischer sporadic simple group $Fi_{23}$. In particular, we compute $(p, q, r)$-generations for the Fischer group $Fi_{23}$, where $p, q$ and $r$ are prime divisors of $|Fi_{23}|$. 
Group generations have played a significant role in solving problems in diverse areas of mathematics such as topology, geometry and number theory.

Generation of a group by its suitable subsets have been the subject of research since the origins of group theory.
A group $G$ is said to be $(l, m, n)$-generated if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = n$.

In such case, $G$ is a quotient group of the Von Dyck group $D(l, m, n)$, and therefore it is also $(\pi(l), \pi(m), \pi(n))$-generated for any $\pi \in S_3$. Thus we may assume throughout that $l \leq m \leq n$.

Further, we emphasize that attention may be restricted to $(p, q, r)$-generations where $p$, $q$, $r$ are primes. Indeed, $(l, m, n)$-generation follows from $(p, q, r)$-generation provided $p = l^\alpha$, $q = m^\beta$, $r = n^\gamma$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$. 
Initially, the study of \((l, m, n)\)-generations of a group \(G\) had deep connections to the topological problem of determining the least genus of an orientable surface on which \(G\) admits an effective, orientation-preserving, conformal action.

In MOORI [13], such investigations were extended well beyond the "minimum genus problem" to all possible \((l, m, n)\)-generations, assuming \(G\) to be finite and non-abelian simple.

Generational results of this type have since proved to be quite useful and interesting.
Groups that can be generated by an involution and an element of order 3 are said to \((2, 3)\)-generated, and such generations have been of particular interest to combinatorists and group theorists.

Any group generated by an involution and an element of order 3 is a quotient group of \(PSL(2, \mathbb{Z})\).

Connections with Hurwitz groups, regular maps, Beauville surfaces and structures provide additional motivation for the study of these groups. (Recall that a Hurwitz group is one that can be \((2, 3, 7)\)-generated.)
If a simple group $G$ is $(l, m, n)$-generated, then by CONDER [5] either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.


**Problem**

Given a non-abelian finite simple group $G$ with $l$, $m$ and $n$ dividing $|G|$ such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Is $G (l, m, n)$-generated?
Many researchers have answered this question since then:

- In a series of articles Prof. Moori (with his research team at Pietermaritzburg):
  \[ HS, McL, J_i \ (1 \leq i \leq 4) \ Co_2, Co_3 \text{ and } Fi_{22} \]

- Prof. Darafsheh and Prof. Ashrafi (with their research teams in Iran):
  \[ Co_1, Th, O'N, Ly, He. \]

- Further, in collaborations with Prof. Moori and Prof. Woldar, we investigated the groups:
  \[ Fi_{23}, Fi'_{24}, \text{ The Baby Monster group } \mathbb{B} \]
In the present talk, we investigate \((p, q, r)\)-generations for the Fischer group \(Fi_{23}\). Since \((2, 3, 3)\)- and \((2, 3, 5)\)-generated groups are quotients of \(A_4\) and \(A_5\) respectively, we need only to consider here the cases when \(r = 7, 11, 13, 17, 23\).
Throughout this article we use the same notation and terminology as can be found in [1, 2, 8, 10, 14]. In particular, for a finite group $G$ with conjugacy classes $C_1$, $C_2$, $C_3$, we denote the corresponding structure constant of $G$ by $\Delta(G) = \Delta_G(C_1, C_2, C_3)$. Observe that $\Delta(G)$ is nothing more than the cardinality of the set $\Omega = \{(x, y) \mid xy = z\}$ where $x \in C_1$, $y \in C_2$ and $z$ is a fixed representative in the conjugacy class $C_3$. It is well known that the value of $\Delta(G)$ can be computed from the character table of $G$ (e.g., see [11, p.45]) via the formula

$$\Delta_G(C_1, C_2, C_3) = \frac{|C_1||C_2|}{|G|} \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\chi_i(g_k)}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \cdots, \chi_m$ are the irreducible complex characters of $G$, and the bar denotes complex conjugation.
We denote by $\Delta^*(G) = \Delta^*_G(C_1, C_2, C_3)$ the number of distinct ordered pairs $(x, y) \in \Omega$ such that $G = \langle x, y \rangle$. Clearly, if $\Delta^*(G) > 0$ then $G$ is $(l, m, n)$-generated where $l, m, n$ are the respective orders of elements from $C_1, C_2, C_3$. In this instance we shall also say that $G$ is $(C_1, C_2, C_3)$-generated and we shall refer to $(C_1, C_2, C_3)$ as a generating triple for $G$.

Further, if $H$ is a subgroup of $G$ containing the fixed element $z \in C_3$ above, we denote by $\Sigma(H) = \Sigma_H(C_1, C_2, C_3)$ the total number of distinct ordered pairs $(x, y) \in \Omega$ such that $\langle x, y \rangle \leq H$. The value of $\Sigma_H(C_1, C_2, C_3)$ is obtained as the sum of all structure constants $\Delta_H(c_1, c_2, c_3)$ where the $c_i$ are conjugacy classes of $H$ that fuse to $C_i$ in $G$, i.e., $c_i \subseteq H \cap C_i$. The number of pairs $(x, y) \in \Omega$ generating a subgroup $H$ of $G$ will be denoted by $\Sigma^*(H) = \Sigma^*_H(C_1, C_2, C_3)$, and the centralizer of a representative of the conjugacy class $C$ by $C_G(C)$. 
A general conjugacy class of a proper subgroup $H$ of $G$ whose elements are of order $n$ will be denoted by $nx$, reserving the notation $nX$ for the case where $H = G$. The number of conjugates of a given subgroup $H$ of $G$ containing a fixed element $g$ is given by $\pi(g)$, where $\pi$ is the permutation character corresponding to the action of $G$ on the cosets of $H$, i.e., $\pi$ is the induced character $(1_H)^G$ ([11]). As the stabilizer of $H$ in this action is clearly $N_G(H)$, in many cases one can more easily compute the value $\pi(g)$ from the fusion map from $N_G(H)$ into $G$ in conjunction with Theorem 4.1 below. We emphasize that this is an especially useful strategy when the decomposition of $\pi$ into irreducible characters is not known explicitly.
Thus, in order to compute $\Delta^*(G)$, we need the character tables of $G$ and character tables of $M_1, M_2, \ldots, M_t$ together with information on $M_i \cap M_j$. However, amongst the maximal subgroups $M_j$ containing $z$, there may be conjugate subgroups. In such situation the following theorem is very helpful.
(GANIEF & MOORI [10]) Let $G$ be a finite group and let $H$ be a subgroup of $G$ containing a fixed element $g$ such that $\gcd(o(g), |N_G(H) : H|) = 1$. Then the number of conjugates of $H$ containing $z$ is given by

$$\pi(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(g_i)|}$$

where $\pi$ is the permutation character corresponding to the action of $G$ on the cosets of $H$, and $g_1, g_2, \ldots, g_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the $G$-class containing $g$. 
Non-Generation

Below we provide some very useful techniques for establishing non-generation.

**Lemma**

(CONDER, WILSON, & WOLDAR [6]) Let $G$ be a finite centerless group and suppose $lX$, $mY$, $nZ$ are $G$-conjugacy classes for which

$$\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(nZ)|.$$ 

Then $\Delta^*(G) = 0$ and therefore $G$ is not $(lX, mY, nZ)$-generated.
Lemma

(CONDER [5]) Suppose $a$ and $b$ are permutations of $N$ points such that $a$ has $\lambda_u$ cycles of length $u$ ($1 \leq u \leq l$) and $b$ has $\mu_v$ cycles of length $v$ ($1 \leq v \leq m$) and their product $ab$ is an involution having $k$ transpositions and $N - 2k$ fixed points. If $a$ and $b$ generate a transitive group on these $N$ points, then there exists a non-negative integer $\alpha$ such that

\[ k = 2\alpha - 2 + \sum_{1 \leq u \leq l} \lambda_u + \sum_{1 \leq v \leq m} \mu_v. \]
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**Definition**

A group $G$ is called a 3-transposition group if it is generated by a conjugacy class $D$ of involutions in $G$ such that $o(de) \leq 3$ for all $d$ and $e$ in $D$. The conjugacy class $D$ is called a class of conjugate 3-transpositions.

- *Fischer* introduced and investigated 3-transposition groups. He classified all finite 3-transposition groups with no non-trivial normal soluble subgroups.
- In the process of classifying the 3-transposition groups, Fischer discovered three new groups $\text{Fi}_{22}$, $\text{Fi}_{23}$ and $\text{Fi}_{24}$ with 3510, 31671 and 306936 transpositions respectively.
- Of these, the first two groups are simple, while the third contains a simple normal subgroup $\text{Fi}'_{24}$ of index 2 (consisting of the products of evenly many transpositions)
(p, q, r)-Generations of $Fi_{23}$

- The Fischer’s sporadic group $Fi_{23}$ has order

$$4089470473293004800 = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \approx 4 \times 10^{18}$$

- The group $Fi_{23}$ has 94 conjugacy classes of its elements in total including three involution classes and four classes of elements of order 3, namely $2A, 2B, 2C, 3A, 3B, 3C$ and $3D$ as represented in ATLAS [7].

- Kleidman, Parker and Wilson [12] classified all the maximal subgroups of $Fi_{23}$. There are 14 conjugacy classes of maximal subgroups of $Fi_{23}$. 
(2, 3, 11)-Generations of $Fi_{23}$

In order to investigate $(p, q, 11)$-generations of $Fi_{23}$ we require knowledge of all the its maximal subgroups with order divisible by 11. They are, up to isomorphisms,

$$2 \cdot Fi_{22}, \quad 2^2 \cdot U_6(2).2, \quad 2^{11} \cdot M_{23}, \quad S_{12}, \quad L_2(23)$$

Lemma

The group $Fi_{23}$ is $(2X, 3Y, 11A)$-generated, for $X \in \{A, B, C\}$ and $Y \in \{A, B, C, D\}$, if and only if the ordered pair $(X, Y) = (C, D)$. 
Proof:

As $Fi_{23}$ has unique class of elements of order 11, we have 12 triples of classes to consider in this case.


For any triple $(2X, 3Y, 11A) \in T$, non-generation follows at once since $\Delta_{Fi_{23}}(2X, 3Y, 11A) = 0$ in those case. Further $(2B, 3C, 11A)$ is not a generating triple for $Fi_{23}$ since

$$\Delta_{Fi_{23}}(2B, 3C, 11A) = 11 < 44 = |C_{Fi_{23}}(11A)|.$$ 

Next, we consider the case $(2B, 3D, 11A)$. 


Case \((2B, 3D, 11A)\)

Let \(L \cong M_{12}\) be contained in the conjugacy class of subgroups with non-empty intersection with the classes \(2B, 3D\) and \(11A\). Observe that \(N_{Fi_{23}}(L) = C_2 \times M_{12}\).

Let \(z \in L\) be a fixed element of order 11. Then the fusion map of \(L\) into \(Fi_{23}\) yields

\[2a \rightarrow 2B, \quad 3a \rightarrow 3D, \quad 11a \rightarrow 11A, \quad 11b \rightarrow 11B.\]

Since \(|C_{C_2 \times M_{12}}(z)| = 22\) and \(|C_{Fi_{23}}(z)| = 44\), it follows that \(z\) is contained in exactly 4 conjugates of \(M_{12}\).

Further, note that no maximal subgroup of \(L\) and hence no proper subgroup of \(L\) is \((2B, 3D, 11A)\)-generated.
We calculate $\Sigma_{M_{12}}(2B, 3D, 11A) = 11 = \Sigma_{C_2 \times M_{12}}(2B, 3D, 11A)$. Therefore

$$\Delta^*_{Fi_{23}}(2B, 3D, 11A) \leq \Delta_{Fi_{23}}(2B, 3D, 11A) - 4 \Sigma_{M_{12}}(2B, 3D, 11A) = 44 - 4(11) = 0,$$

showing that $Fi_{23}$ is not $(2B, 3D, 11A)$-generated.
Next we examine the triple \((2C, 3C, 11A)\). For this we consider the transitive action of the group \(Fi_{23}\) on the cosets of \(2.Fi_{22}\) with permutation character

\[
\pi = 1a + 782a + 30888a
\]

(see [7]). Recall that the value of \(\pi(g), g \in Fi_{23}\), is the number of cosets of \(Fi_{23}\) fixed by \(g\). Set \(N = |Fi_{23} : 2.Fi_{22}| = 31671\). Referring to Lemma 4.3, we have

\[
\begin{align*}
\lambda_3 &= \frac{N-135}{3} = 10512 \\
\mu_{11} &= \frac{N-2}{11} = 2879 \\
k &= \frac{N-183}{2} = 15744
\end{align*}
\]

from which we get a contradiction since \(\alpha = \frac{2355}{2} \not\in \mathbb{Z}\). Thus \(Fi_{23}\) is not \((2C, 3C, 11A)\)-generated.
Finally, we consider the triple $(2C, 3D, 11A)$. We calculate the structure constant $\Delta_{Fi_{23}}(2C, 3D, 11A) = 11616$.

The maximal subgroups of $Fi_{23}$ with order divisible by 11, up to automorphisms, are $2.Fi_{22}$, $2^2.U_6(2).2$, $2^{11}.M_{23}$, $S_{12}$ and $L_2(23)$. However, the maximal subgroups $2^2.U_6(2).2$, and $2^{11}.M_{23}$ does not meet the $Fi_{23}$-conjugacy class $3D$. That is, $3D \cap 2^2.U_6(2).2 = \emptyset = 2^{11}.M_{23} \cap 3D$.

Further, a fixed element $z$ of order 11 is contained in two conjugate copies of $2.Fi_{22}$, four copies of $S_{12}$ and 20 copies of $L_2(23)$. By looking at the fusion maps from three maximal subgroups into the Fischer group $Fi_{23}$, we calculate
\[ \Delta_{Fi_{23}}^* (2C, 3D, 11A) \geq \Delta(Fi_{23}) - 2 \sum(2.Fi_{22}) - 4 \sum(S_{12}) - 20 \sum(L_2(23)) = 11616 - 2(1980) - 4(110) - 20(22) = 6776, \]

and generation of \( Fi_{23} \) follows by the triple \((2C, 3D, 11A)\). This completes the proof. \( \blacksquare \)
By using similar techniques, we compute generating pairs for the Fischer group $Fi_{23}$. We summarize our results in the form following theorem:

**Theorem**

The Fischer group $Fi_{23}$ is $(p, q, r)$-generated for all $p, q, r \in \{2, 3, 5, 7, 11, 17, 23\}$ with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$ or $(p, q, r) = (2, 3, 7)$.
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Thank you for your presence !!!!
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