Certain Monomial Characters and Their Subnormal Constituents

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This is a joint work with G. Navarro.
Introduction
Let $G$ be a group. A character $\chi \in \text{Irr}(G)$ is said to be \textbf{monomial} if there exist a subgroup $U \subseteq G$ and a linear $\lambda \in \text{Irr}(U)$, such that

$$\chi = \lambda^G.$$
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A group $G$ is said to be \textit{monomial} if all its irreducible characters are monomial.
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Thus, supersolvable groups are monomial groups. But this result depends more on the structure of the group than on characters themselves.
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**Theorem (Gow)**

Let $G$ be a solvable group. Suppose that $\chi \in \text{Irr}(G)$ takes real values and has odd degree. Then $\chi$ is rational-valued and monomial.
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We give a monomiality criterium which also deals with fields of values and degrees of characters.
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**Theorem A**

Let \( G \) be a \( p \)-solvable group. Assume that \( |N_G(P) : P| \) is odd, where \( P \in \text{Syl}_p(G) \) for some prime \( p \). If \( \chi \in \text{Irr}(G) \) has degree not divisible by \( p \) and the values of \( \chi \) are contained in the cyclotomic extension \( \mathbb{Q}|_{G|_p} \), then \( \chi \) is monomial.
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When \( p = 2 \), we can recover Gow’s result from Theorem A.
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$B_\pi$ Theory
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A $B_\pi$ character of a group $G$ may be thought as an irreducible character of $G$ induced from a $\pi$-special character of some subgroup of $G$. (True in groups of odd order).
Main results
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Theorem B

Let $G$ be a $p$-solvable group. Assume that $|N_G(P) : P|$ is odd, where $P \in \text{Syl}_p(G)$ for some prime $p$. If $\chi \in \text{Irr}(G)$ has degree not divisible by $p$ and its values are contained in the cyclotomic extension $\mathbb{Q}|_{G|_p}$, then $\chi$ is a $B_p$ character of $G$. 

Notice that $B_p$ characters with degree not divisible by $p$ are monomial. Thus Theorem B implies Theorem A.
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**Corollary C**

Let $G$ be a $p$-solvable group. Suppose that $|N_G(P) : P|$ is odd, where $P \in \text{Syl}_p(G)$ for some prime $p$. If $\chi \in \text{Irr}(G)$ has degree not divisible by $p$ and its field of values is contained in $\mathbb{Q}|G|_p$, then every subnormal constituent of $\chi$ is monomial.
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**Corollary C**

Let $G$ be a $p$-solvable group. Suppose that $|N_G(P):P|$ is odd, where $P \in \text{Syl}_p(G)$ for some prime $p$. If $\chi \in \text{Irr}(G)$ has degree not divisible by $p$ and its field of values is contained in $\mathbb{Q}_{|G|_p}$, then every subnormal constituent of $\chi$ is monomial.

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**Corollary D**

Let $G$ be a $p$-solvable group. Assume that $|N_G(P) : P|$ is odd, where $P \in Syl_p(G)$ for some prime $p$. The number of irreducible characters which have degree not divisible by $p$ and field of values contained in $\mathbb{Q}_{|G|_p}$ equals the number of orbits under the natural action of $N_G(P)$ on $P/P'$.

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The number of such characters can be computed locally.
Thanks for your attention!