Recent Results on Generalized Baumslag-Solitar Groups

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Baumslag-Solitar groups

(i) A *Baumslag-Solitar group* is a group with a presentation

\[
BS(m, n) = \langle t, x \mid (x^m)^t = x^n \rangle,
\]

where \( m, n \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\} \).

(ii) A similar type of 1-relator group is

\[
K(m, n) = \langle x, y \mid x^m = y^n \rangle,
\]

where \( m, n \in \mathbb{Z}^* \).

These are the fundamental groups of certain graphs of groups.
Let $\Gamma$ be a finite connected graph. For each edge $e$ label the endpoints $e^+$ and $e^-$. Infinite cyclic groups $\langle g_x \rangle$ and $\langle u_e \rangle$ are assigned to each vertex $x$ and edge $e$.

Injective homomorphisms $\langle u_e \rangle \rightarrow \langle g_{e^+} \rangle$ and $\langle u_e \rangle \rightarrow \langle g_{e^-} \rangle$ are defined by

$$u_e \mapsto g_{e^+}^{\omega^+(e)} \quad \text{and} \quad u_e \mapsto g_{e^-}^{\omega^-(e)}$$

where $\omega^+(e), \omega^-(e) \in \mathbb{Z}^*$. 
So we have a weight function

\[ \omega : E(\Gamma) \rightarrow \mathbb{Z}^* \times \mathbb{Z}^* \]

where \( \omega(e) = (\omega^-(e), \omega^+(e)) \) is defined up to \( \pm \). The weighted graph

\[ (\Gamma, \omega) \]

is a generalized Baumslag-Solitar graph or GBS-graph.
The generalized Baumslag-Solitar group (GBS-group) determined by the GBS-graph \((\Gamma, \omega)\) is the fundamental group \(G = \pi_1(\Gamma, \omega)\). If \(T\) is a maximal subtree of \(\Gamma\), then \(G\) has generators \(g_x\) and \(t_e\), with relations

\[
\begin{align*}
\left( g_{e^+}^{\omega^+(e)} \right) t_e &=\quad g_{e^-}^{\omega^-(e)}, \quad \text{for } e \in E(T), \\
\left( g_{e^+}^{\omega^+(e)} \right) &=\quad g_{e^-}^{\omega^-(e)}, \quad \text{for } e \in E(\Gamma) \setminus E(T).
\end{align*}
\]

If \(\Gamma\) is an edge \(e\), \(G = K(m, n)\): if \(\Gamma\) is a single loop \(e\), \(G = BS(m, n)\), where \(m = \omega^+(e)\), \(n = \omega^-(e)\).
The maximal subtree $T$ is the path $x, y, z, u$. The GBS-group has a presentation in $r, s, t, g_x, g_y, g_z, g_u$ with relations

\[
g_x^2 = g_y^{-3}, \quad g_y^4 = g_z^4, \quad g_z^5 = g_u^3
\]
\[
(g_x^2)^r = g_x^2, \quad (g_x^4)^s = g_y^{-1}, \quad (g_u^{12})^t = g_y^{20}.
\]
Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group.

(i) $G$ is independent of the choice of maximal subtree.

(ii) $G$ is finitely presented and torsion-free.

(iii) If $\Gamma$ is a tree, then $G$ is residually finite and hence is hopfian.

The next result is due to P. Kropholler.

(iv) The non-cyclic GBS-groups are exactly the finitely generated groups of cohomological dimension 2 which have a commensurable infinite cyclic subgroup.
(v) If $H$ is a finitely generated subgroup of a GBS-group $G$, either $H$ is a GBS-group or it is free. Hence $G$ is coherent.

Proof. We have $\text{cd}(H) \leq \text{cd}(G) = 2$. If $\text{cd}(H) = 1$, then $H$ is free by the Stallings-Swan Theorem. Otherwise $\text{cd}(H) = 2$. If $H$ contains a commensurable element, it is a GBS-group by (iv). If $H$ has no commensurable elements, it is free.

(vi) The second derived subgroup of a GBS-group is free. (Kropholler.)
The weight of a path

Let \((\Gamma, \omega)\) be a GBS-graph with a maximal subtree \(T\). Let \(e = \langle x, y \rangle\) be a non-tree edge where \(x \neq y\). There is a unique path in \(T\) from \(x\) to \(y\), say

\[ x = x_0, x_1, \ldots, x_n = y. \]

Then there is a relation in \(G = \pi_1(\Gamma, \omega)\)

\[ g_x^{p_1(e)} = g_y^{p_2(e)} \]

where \(p_1(e)\) and \(p_2(e)\) are the products of the left and right weight values of the edges in the tree path \([x, y]\).
Lemma 1. Let $(\Gamma, \omega)$ be a GBS-graph with a maximal subtree $T$. Let $\alpha = [x, y]$ be a path in $T$. Then there exist $a, b \in \mathbb{Z}^*$ such that $g_x^a = g_y^b$ in $\pi_1(\Gamma, \omega)$. Also, if $g_x^m = g_y^n$, then $(m, n) = (a, b)q$ for some $q \in \mathbb{Z}^*$.

Definition. Call $(a, b)$ the weight of the path $\alpha$ in $T$ and denote it by $\omega_T(\alpha)$ or

$$\omega_T(x, y) = (\omega_T^{(1)}(x, y), \omega_T^{(2)}(x, y)).$$

This is unique up to $\pm$. 
How to compute the weight of a path

Let $\alpha$ be the path $x = x_0, x_1, \ldots, x_n = y$ and write $\omega(\langle x_i, x_{i+1} \rangle) = (u_i^{(1)}, u_i^{(2)}), \ i = 0, 1, \ldots, n - 1$. Define $(\ell_i, m_i), \ 0 \leq i \leq n$, recursively by $\ell_0 = 1 = m_0$ and

$$\ell_{i+1} = \frac{\ell_i u_i^{(1)}}{\gcd(m_i, u_i^{(1)})}, \quad m_{i+1} = \frac{m_i u_i^{(2)}}{\gcd(m_i, u_i^{(1)})}.$$  

Then

**Lemma 2.** $\omega_T(x, y) = (\ell_n, m_n)$. 
Tree and skew tree dependence

Let \((\Gamma, \omega)\) be a GBS-graph with a maximal subtree \(T\). The non-tree edge \(e = \langle x, y \rangle\) is called \(T\)-dependent or skew \(T\)-dependent if and only if

\[
\frac{\omega^-(e)}{\omega^+(e)} = \frac{\omega_T^{(1)}(e)}{\omega_T^{(2)}(e)} \quad \text{or} \quad -\frac{\omega_T^{(1)}(e)}{\omega_T^{(2)}(e)}
\]

respectively. If \(e\) is a loop, then \(e\) is \(T\)-dependent (skew \(T\)-dependent) if and only if \(\omega^-(e) = \omega^+(e)\) or \(\omega^-(e) = -\omega^+(e)\) respectively.
If every non-tree edge of a GBS-graph is $T$-dependent, the GBS-graph is called \emph{tree dependent}.

If every non-tree edge is $T$-dependent or skew $T$-dependent with at least one of the latter, then the GBS-graph is called \emph{skew tree dependent}.

These properties are independent of the choice of $T$.

Tree dependence is relevant to the computation of homology in low dimensions.
Theorem 1. (DR). Let \( G = \pi_1(\Gamma, \omega) \) be a GBS-group. Then the torsion-free rank of \( H_1(G) = G_{ab} \) is

\[
r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 + \epsilon
\]

where \( \epsilon = 1 \) if \((\Gamma, \omega)\) is tree dependent and otherwise \( \epsilon = 0 \). Hence tree dependence is independent of the choice of maximal subtree.

Theorem 2. (DR). For any GBS-group \( G \) the Schur multiplier \( H_2(G) \) is free abelian of rank \( r_0(G) - 1 \).
Let $G$ be a group with a commensurable element $x$ of infinite order. If $g \in G$, then $\langle x \rangle \cap \langle x \rangle^g \neq 1$ and $(x^n)^g = x^m$ for $m, n \in \mathbb{Z}^*$. Define $\Delta_x(g) = \frac{m}{n}$. Then

$$\Delta_x : G \rightarrow \mathbb{Q}^*$$

is a well defined homomorphism.

If $y \in G$ is commensurable and $\langle x \rangle \cap \langle y \rangle \neq 1$, then $\Delta_x = \Delta_y$. If this holds for all commensurable elements, then $\Delta_x$ depends only on $G$: denote it by

$$\Delta^G.$$
The $\Delta$-function of a GBS-group

A GBS-graph $(\Gamma, \omega)$ or the group $G = \pi_1(\Gamma, \omega)$, is called \textit{elementary} if $G \simeq BS(1, \pm 1)$. If $G$ is non-elementary, then each commensurable element of $G$ is elliptic and hence is conjugate to a power of some $g_v$. Hence $\Delta^G$ is unique.

\textbf{Lemma 3.} Let $(\Gamma, \omega)$ be a non-elementary GBS-graph, with $T$ a maximal subtree, and let $G = \pi_1(\Gamma, \omega)$. Then:

(i) $\Delta^G(g_v) = 1$ for all $v \in V(\Gamma)$;

(ii) If $e \in E(\Gamma) \setminus E(T)$, $\omega(e) = (a, b)$, $\omega_T(e) = (m, n)$,

$$\Delta^G(t_e) = \frac{an}{bm}.$$
Corollary. (G. Levitt). Let $e$ be a non-tree edge. Then:

(i) $e$ is $T$-dependent if and only if $\Delta^G(t_e) = 1$. Hence $(\Gamma, \omega)$ is tree dependent if and only if $\Delta^G$ is trivial.

(ii) $e$ is skew $T$-dependent if and only if $\Delta^G(t_e) = -1$. Hence $(\Gamma, \omega)$ is skew tree dependent if and only if $\text{Im}(\Delta^G) = \{\pm 1\}$.

If $\text{Im}(\Delta^G) \subseteq \{\pm 1\}$, call $G = \pi_1(\Gamma, \omega)$ unimodular.
The following result tells us when the center of a GBS-group is non-trivial.

**Theorem 3.** Let $(\Gamma, \omega)$ be a GBS-graph and let $G$ be its fundamental group. Assume that $G$ is non-elementary. Then the following are equivalent.

(a) $Z(G)$ is non-trivial.
(b) $\Delta^G$ is trivial.
(c) $(\Gamma, \omega)$ is tree-dependent.
Let \((\Gamma, \omega)\) be a GBS-graph. In finding \(Z(G)\) we may assume the graph is non-elementary. We can also assume \((\Gamma, \omega)\) is tree dependent since otherwise \(Z(G) = 1\).

In a GBS-graph the \textit{distal weight} of a leaf in a maximal subtree is the weight occurring at the vertex of degree 1. In finding the centre there is no loss in assuming there are no leaves with distal weight \(\pm 1\).
Lemma 4. Let \((\Gamma, \omega)\) be a non-elementary GBS-graph with a maximal subtree \(T\). Assume no leaves of \(T\) have distal weight \(\pm 1\). Then

\[
Z(G) \leq \bigcap_{x \in V(\Gamma)} \langle g_x \rangle.
\]

For any \(x, v \in V(\Gamma)\), \(\langle g_x \rangle \cap \langle g_v \rangle = \langle g_{v}^{\omega_{T}^{(1)}(v,x)} \rangle\). Hence

\[
\bigcap_{x \in V(\Gamma)} \langle g_x \rangle = \langle g_{v}^{h_v} \rangle
\]
where

\[
h_v = \text{lcm}\{\omega_{T}^{(1)}(v,x) \mid x \in V(\Gamma)\} = \omega_{T}^{\text{tot}}(v),
\]
the total weight of \(v\) in \(T\).
Locating the centre

The total weight of \( v \) in \( T \) is the smallest positive power of \( g_v \) belonging to every vertex subgroup.

There is a more economic expression for the total weight. Let \( y_1, y_2, \ldots, y_k \) be the vertices of degree 1 in \( T \). Then

\[
\omega_T^{tot}(v) = \text{lcm}\{\omega_T^{(1)}(v, y_i) \mid i = 1, 2, \ldots, m\}.
\]
Lemma 5. Let \((\Gamma, \omega)\) be a non-elementary GBS-graph with maximal subtree \(T\). Assume that no leaf in \(T\) has distal weight \(\pm 1\). Then
\[
Z(G) = \bigcap_{e \in E(\Gamma) \setminus E(T)} C_J(t_e),
\]
where \(J = \bigcap_{x \in V(\Gamma)} \langle g_x \rangle\). If \(\Gamma = T\), then \(Z(G) = J\).

The centralizers in this formula can be found using:

Lemma 6. Let \(e = \langle x, y \rangle \in E(\Gamma) \setminus E(T)\) be \(T\)-dependent and let \(\omega(e) = (m, n)\) and \(\omega_T(x, y) = (a, b)\). Then
\[
C_{\langle g_x \rangle}(t_e) = \langle g_x^{\text{lcm}(a,m)} \rangle.
\]
Theorem 4. (A. Delgado, DR, M. Timm.) Let $(\Gamma, \omega)$ be a non-elementary, tree dependent GBS-graph with a maximal subtree $T$. Assume no leaf in $T$ has distal weight $\pm 1$. Let $v$ be any fixed vertex and let the non-tree edges be $e_i = \langle x_i, y_i \rangle$, $i = 1, 2, \ldots, k$. Put $\omega(e_i) = (m_i, n_i)$, $\omega_T(x_i, y_i) = (a_i, b_i)$, $\omega_T(v, x_i) = (c_i, d_i)$, and $\ell_i = \text{lcm}(a_i, m_i)$. Then $Z(G) = \langle g_v f_v \rangle$ where

$$f_v = \text{lcm}\left\{ \frac{c_i \ell_i}{\gcd(\ell_i, d_i)} \mid i = 1, 2, \ldots, k \right\}.$$

Call $f_v = \omega_{\Gamma}^{\text{tot}}(v)$, the total weight of $v$ in $(\Gamma, \omega)$. 
The two non-tree edges are $e_1, e_2$ and $v$ is the root of the maximal subtree $T$, while $y_1, y_2, y_3$ are the vertices of degree 1 in $T$. The edges $e_1$ and $e_2$ are $T$-dependent, so $(\Gamma, \omega)$ is tree dependent and $Z(G) \neq 1$. 
Read off the required data from the GBS-graph.

\[ \omega^{\text{tot}}_T(v) = \text{lcm}(\omega^{(1)}_T(v, y_1), \omega^{(1)}_T(v, y_2), \omega^{(1)}_T(v, y_3)) = 210. \]

Next

\[ (m_1, n_1) = \omega(e_1) = (35, 8), \quad (m_2, n_2) = \omega(e_2) = (18, 27), \]

\[ (a_1, b_1) = \omega_T(y_1, y_2) = (35, 8), \quad (a_2, b_2) = \omega_T(x, z) = (2, 3), \quad (c_1, d_1) = \omega_T(v, y_1) = (6, 5), \quad (c_2, d_2) = \omega_T(v, x) = (3, 2). \]

Hence \( \ell_1 = 35, \ell_2 = 18 \) and \( \omega^{\text{tot}}_T(v) = 1890. \) Therefore

\[ Z(G) = \langle g_v^{1890} \rangle. \]
In skew tree dependent GBS-graphs the role of the centre is played by the unique maximum normal cyclic subgroup.

**Lemma 7.** Let \((\Gamma, \omega)\) be a non-elementary GBS-graph. Then \(G = \pi_1(\Gamma, \omega)\) has a unique maximal cyclic normal subgroup \(C(G)\).

**Proof.** Suppose \(\{C_i|i \in I\}\) is an infinite ascending chain of cyclic normal subgroups of \(G\). Each \(C_i\) is commensurable and hence lies in a vertex subgroup. Hence infinitely many of the \(C_i\) lie in some \(\langle g_v \rangle\), a contradiction. Hence \(G\) has a maximal cyclic normal subgroup \(C\).
It is straightforward to show $C$ is unique.

**Corollary.** $C(G) \leq \bigcap_{v \in V(\Gamma)} \langle g_v \rangle = J$ and hence

$$C(G) = \bigcap_{e \in E(\Gamma) \setminus E(T)} J_{\langle t_e \rangle},$$

where $J_{\langle t_e \rangle}$ is the $\langle t_e \rangle$-core of $J$.

The subgroup $C(G)$ in a GBS-group can be trivial.
Lemma 8. Let $G = \pi_1(\Gamma, \omega)$ be a non-elementary GBS-group. Then:

(i) $C(G) \neq 1$ if and only if $\pi_1(\Gamma, \omega)$ is unimodular, i.e., $(\Gamma, \omega)$ is either tree dependent or skew-tree dependent.

(ii) $1 = Z(G) < C(G)$ if and only if $(\Gamma, \omega)$ is skew tree dependent.
The algorithm to compute the centre of a tree dependent GBS-graph can be applied to a skew tree dependent GBS-graph \((\Gamma, \omega)\), with cores playing the role of centralizers. It will then compute \(C(\pi_1(\Gamma, \omega))\).

**Theorem 5.** (A. Delgado, DR, M. Timm.) *Let \((\Gamma, \omega)\) be a non-elementary, skew tree dependent GBS-graph with a maximal subtree \(T\) having no distal weights \(\pm 1\). Then if \(G = \pi_1(\Gamma, \omega)\) and \(v\) is any vertex \(v\),

\[
C(G) = \langle g_v^{\omega_{\Gamma}^{\text{tot}}(v)} \rangle.
\]
An example

Change the weight of edge $e_1$ in the last example from $(35, 8)$ to $(35, -8)$.

$$Z(G) = 1, \quad C(G) = \langle g_v^{\omega_{\text{tot}}(v)} \rangle = \langle g_v^{1890} \rangle.$$
What is the relation between GBS-groups and 3-manifold groups, i.e., the fundamental groups of compact 3-manifolds?

Some examples (W. Heil).

1. \( K(m, n) = \langle x, y \mid x^m = y^n \rangle \) is a 3-manifold group.

2. The group \( \langle x_1, x_2, x_3 \mid x_1^m = x_2^n, \ x_2^m = x_3^n \rangle \) is a 3-manifold group iff \(|m| = 1\) or \(|n| = 1\) or \(|m| = |n|\).
GBS-groups and 3-manifold groups

3. \( B(m, n) = \langle t, x \mid x^n = (x^m)^t \rangle \) is a 3-manifold group iff \( |m| = |n| \).

*Problem.* Find necessary and sufficient conditions on a GBS-graph \((\Gamma, \omega)\) for \(\pi_1(\Gamma, \omega)\) to be the fundamental group of a compact 3-manifold.

A GBS-graph \((\Gamma, \omega)\) is called *locally weight constant* if at every vertex \(v\) all weights equal \(c_v\) and *locally \(\pm\) weight constant* if all weights at \(v\) equal \(\pm c_v\) for some constant \(c_v\).
Remarks
Let \((\Gamma, \omega)\) be a GBS-graph. If \((\Gamma, \omega)\) is locally weight constant GBS-graph, it is tree dependent. If it is locally \(\pm\) weight constant, it is tree or skew tree dependent, i.e., it is unimodular.

Example  The GBS-graph shown is locally \(\pm\) weight constant, but not locally weight constant.
The GBS-groups which are 3-manifold groups

**Theorem 6.** (A. Delgado, DR, M. Timm.) Let \((\Gamma, \omega)\) be a non-elementary GBS-graph. Then the following are equivalent.

(i) \(\pi_1(\Gamma, \omega)\) is a 3-manifold group.

(ii) \(\pi_1(\Gamma, \omega)\) is an orientable 3-manifold group.

(iii) \((\Gamma, \omega)\) is locally \(\pm\) weight constant.

This explains Heil’s examples: \(B(m, n)\) is a 3-manifold group if and only if \(|m| = |n|\).
3-manifold GBS-group covers

Let \((\Gamma, \omega)\) be a non-elementary GBS-graph. If \(\pi_1(\Gamma, \omega)\) is not a 3-manifold group, it may be a quotient of a GBS-group which is a 3-manifold group.

A 3-manifold GBS-group cover of \(\pi_1(\Gamma, \omega)\) is a surjective homomorphism

\[ \varphi : \pi_1(\Gamma, \tau) \rightarrow \pi_1(\Gamma, \omega) \]

where \((\Gamma, \tau)\) is a GBS-graph such that \(\pi_1(\Gamma, \tau)\) is a 3-manifold group, and \(\varphi\) is a pinch map, which arises by dividing the weights on certain edges of \(\Gamma\) by common factors.
Theorem 7. (A. Delgado, DR, M. Timm.) Let \((\Gamma, \omega)\) be a non-elementary GBS-graph. Then the following are equivalent.

(i) \(\pi_1(\Gamma, \omega)\) has a 3-manifold GBS-group cover.

(ii) \(\pi_1(\Gamma, \omega)\) has an orientable 3-manifold GBS-group cover.

(iii) \(\pi_1(\Gamma, \omega)\) is unimodular, i.e., \((\Gamma, \omega)\) is tree dependent or skew tree dependent.
Suppose that \((\Gamma, \omega)\) is a non-elementary GBS-graph such that \(\pi_1(\Gamma, \omega)\) unimodular. We show how to construct a 3-manifold GBS-group cover of \(\pi_1(\Gamma, \omega)\).

**Case:** \((\Gamma, \omega)\) is tree dependent.

Define a new weight function \(\tau\) on \(\Gamma\) as follows:

\[
\tau(e) = (\omega_{\Gamma}^{\text{tot}}(e^-), \omega_{\Gamma}^{\text{tot}}(e^+)), \quad e \in E(\Gamma).
\]

Call the GBS-graph \((\Gamma, \tau)\) the **total weight cover** of \((\Gamma, \omega)\).
Clearly the total weight cover is locally weight constant, so $\pi_1(\Gamma, \tau)$ is a compact (orientable) 3-manifold group.

The identity map on $\Gamma$ and a suitable sequence of pinches yields a surjective homomorphism

$$\varphi : \pi_1(\Gamma, \tau) \to \pi_1(\Gamma, \omega)$$

which is a 3-manifold GBS-group cover of $\pi_1(\Gamma, \omega)$. 
The total ± weight cover

**Case:** \((\Gamma, \omega)\) is skew tree dependent

Let \(T\) be a maximal subtree in \(\Gamma\). We can assume that all weights in \(T\) are positive. Write \(E(\Gamma) \setminus E(T) = P \cup N\) where \(P\) is the set of edges with positive weights and \(N\) is the set of remaining edges. Define a new weight function \(\tau\) on \(\Gamma\) by

\[
\tau(e) = (\omega^\text{tot}_{\Gamma}(e^-), \omega^\text{tot}_{\Gamma}(e^+)), \quad e \in E(T) \cup P,
\]

and

\[
\tau(e) = (\omega^\text{tot}_{\Gamma}(e^-), -\omega^\text{tot}_{\Gamma}(e^+)), \quad e \in N.
\]
Then \((\Gamma, \tau)\) is a locally \(\pm\) weight constant GBS-graph, the total \(\pm\) weight cover of \((\Gamma, \omega)\). Thus \(\pi_1(\Gamma, \tau)\) is a 3-manifold group and we have a 3-manifold GBS-group cover

\[
\varphi : \pi_1(\Gamma, \tau) \to \pi_1(\Gamma, \omega)
\]
defined by the identity map on \(\Gamma\) and suitable pinches.

**Final comments**

(i) The 3-manifold GBS-group covers constructed are *minimal* in the sense that all others factor through them.

(ii) The kernels of the covering maps can be computed.