Embeddings into Thompson’s group V and coCF groups

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Finite state automaton

A (deterministic) finite state automaton is a quintuple $(S, A, \mu, Y, s_0)$, where

- $S$ is a finite set, called the state set,
- $A$ is a finite set, called the alphabet,
- $\mu : A \times S \rightarrow S$ is a function, called the transition function,
- $Y$ is a (possibly empty) subset of $S$ called the subset of accept states,
- $s_0 \in S$ is called the start state.

If we allow $\mu : A \times S \rightarrow P(S)$, we call it a non-deterministic FSA.

Remark: Every NDFSA is equivalent to a DFSA.
Finite state automaton

Mathematical model of computation to design computer programs.
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Every NDFSA is equivalent to a DFSA.
Finite state automaton: an example
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Regular language and finite groups

Definition
Given a FSA a regular language is the language given by all paths inside the FSA which begin at the start state and end at an accept state.

Definition
Given a finitely generated group \( G = \langle X | R \rangle \), the language of the word problem is \( WP(G) = \{ \text{words } w \text{ in the monoid of } X \cup X^{-1} \text{ such that } w \equiv G^1 \} \).

Theorem (Anisimov)
A finitely generated group \( G \) is finite if and only if \( WP(G) \) is a regular language.
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Pushdown automaton

Definition (PDA - Handwaving)

A PDA is like a FSA but it also employs a stack in its transition function. The transition function pops and pushes a symbol at the top of the stack and uses it to decide which state to reach.

Remark

PDA adds the stack as a parameter for choice. Finite state machines just look at the input signal and the current state: they have no stack to work with.

It is not true that all NDPDA are equivalent to DPDA.
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Pushdown automaton: an example
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Context free languages and virtually free groups

Definition

Context-free languages are those accepted by PDAs.

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A finitely generated group is a context-free group if WP(G) is a context-free language.

Theorem (Muller-Schupp)

A finitely generated group $G$ is virtually free if and only if it is context-free.
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Co-context-free (Co-CF) groups

Definition
Given a finitely generated group $G = \langle X | R \rangle$, the language of the coword problem is $\text{coWP}(G) = \{\text{words } w \text{ in the monoid of } X \cup X^{-1} \text{ such that } w \not\equiv G \}$. 

Definition
A finitely generated group is a co-context-free group (coCF) if $\text{coWP}(G)$ is a context-free language. Let $\text{coCF}$ be the class of all coCF groups.

Remark
Every CF group is in $\text{coCF}$ (the converse is not true).
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A finitely generated group is a co-context-free group (coCF) if $\text{coWP}(G)$ is a context-free language. Let $\text{coC}\mathcal{F}$ be the class of all coCF groups.

Remark
Every CF group is in $\text{coC}\mathcal{F}$ (the converse is not true).
Closure properties of the class $\text{coCF}$

Theorem (Holt-R"over-Rees-Thomas)

The class $\text{coCF}$ is closed under taking

- taking finite direct products,
- taking restricted standard wreath products with context-free top groups,
- passing to finitely generated subgroups,
- passing to finite index overgroups.

Conjecture (Holt-R"over-Rees-Thomas)

$\text{coCF}$ is not closed for free products.

Candidate: $\mathbb{Z} \ast \mathbb{Z}/2$.

Theorem (Bleak-Salazar)

$\mathbb{Z} \ast \mathbb{Z}/2$ does not embed into Thompson’s group $V$. 

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Theorem (Bleak-Salazar)

$\mathbb{Z} \ast \mathbb{Z}^2$ does not embed into Thompson’s group $V$. 
Thompson’s group $F$ is the group $PL_2(I)$, with respect to composition, of all piecewise-linear homeomorphisms of the unit interval $I = [0,1]$ with a finite number of breakpoints, such that

$\triangleright$ all slopes are integral powers of 2,

$\triangleright$ all breakpoints have dyadic rational coordinates.
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1 2 5 3
34 4 2
1 5
1 234
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The group $\mathbb{Q}\text{Aut}(T_{2,c})$ is the infinite binary 2-colored tree (left = red, right = blue).

Definition $\mathbb{Q}\text{Aut}(T_{2,c})$ is the group of all maps $T_{2,c} \rightarrow T_{2,c}$ which respect the edge and color relation, except for possibly finitely many vertices.
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Lehnert’s conjecture
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**Theorem (Lehnert)**

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\[ \text{QAut}(\mathcal{T}_2, c) \text{ is in } \text{coC}\mathcal{F}. \]

Conjecture (Lehnert)
\[ \text{QAut}(\mathcal{T}_2, c) \text{ is a universal coC}\mathcal{F} \text{ group.} \]
The relation between $\mathcal{V}$ and $\mathcal{QAut}(\mathcal{T}_{2,c})$

Our version of his proposed embedding:

1. Given a tree $\mathcal{T}$, regard it as a subtree of $\mathcal{T}_{2,c}$ with root 0 (left child of the root of $\mathcal{T}_{2,c}$).
2. Define a bijection $\omega_{\mathcal{T}}:\{\text{leaves of } \mathcal{T}\} \to \{\text{nodes of } \mathcal{T}_{2,c}\} \cup \{\varepsilon\}$ in the left-to-right order so the rightmost leaf goes to $\varepsilon$.
3. Given $(D,R,\sigma) \in \mathcal{V}$ define its image this way:
   1. $\sigma$ takes subtrees of $\mathcal{T}_{2,c}$ at leaves $D$ to those at leaves of $R$.
   2. If $n$ is a node of $D$ or the root of $\mathcal{T}_{2,c}$, send it to $n \omega_{\mathcal{T}}^{-1} D \sigma \omega_{\mathcal{T}} R$.

Corollary (Lehnert-Schweitzer)
Thompson’s group $\mathcal{V}$ is in $\text{co}\text{CF}$.
The relation between $V$ and $\mathbb{Q}\text{Aut}(\mathcal{T}_{2,c})$

**Theorem (Lehnert)**

$V \hookrightarrow \mathbb{Q}\text{Aut}(\mathcal{T}_{2,c})$.

**Corollary (Lehnert-Schweitzer)**

Thompson's group $V$ is in $\text{co}\mathcal{C}\mathcal{F}$.
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The relation between $V$ and $Q\text{Aut}(\mathcal{T}_{2,c})$

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Thompson's group $V$ is in coCF.
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**Corollary (Lehnert-Schweitzer)**

*Thompson’s group $V$ is in $\text{coC}$F.*
The relation between $V$ and $\mathcal{QAut}(\mathcal{T}_{2,c})$

Lemma (Lehnert, Bleak-M-Neunhöffer)

If $\tau \in \mathcal{QAut}(\mathcal{T}_{2,c})$ there is a pair $d_\tau = (v_\tau, p_\tau)$ representing $\tau$ such that $v_\tau \in V$ acts like $\tau$ beneath a suitable level ($V$-part), $p_\tau$ is a bijection on the nodes above (bijection part).

We call $d_\tau$ a disjoint decomposition.

There are many disjoint decompositions, but we can always define a minimal one (in some sense).
The relation between $V$ and $Q\text{Aut}(T_{2,c})$

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**Lemma (Lehnert, Bleak-M-Neunhöffer)**

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- $v_{\tau} \in V$ acts like $\tau$ beneath a suitable level (V-part),
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There are many disjoint decompositions, but we can always define a minimal one (in some sense).
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Question (Lehnert-Schweitzer)

Does $\mathcal{QAut}(\mathcal{T}_{2,c})$ embed into $V$?

Theorem (Bleak-M-Neunh"offer)

Yes.

Idea of the embedding: start with $\tau \in \mathcal{QAut}(\mathcal{T}_{2,c})$:

$\triangleright$ Build $d_\tau = (v_\tau, p_\tau)$ with $v_\tau = (D_\tau, R_\tau, \sigma_\tau)$,

$\triangleright$ Build a new tree pair $(\hat{D}_\tau, \hat{R}_\tau, \hat{\sigma}_\tau)$ by "expanding $v_\tau$" suitably.
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The relation between $V$ and $\mathbb{Q}Aut(\mathcal{T}_{2,c})$
The relation between $V$ and $QAut(T_{2,c})$

- Replace every node $w$ in $D_{d_\tau}$ by a caret $(w, w_n, w_p)$,
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Node with address $w$...

... becomes a caret in tree for $V$ element.

(But, not at address $w$!)

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\[ W_n \quad W_p \]
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- If $e_{\text{parent}}, e_{\text{left}}, e_{\text{right}}$ are the edges attached to $w$, attach $e_{\text{left}}$ and $e_{\text{right}}$ to the bottom of $w_n$ and $e_{\text{parent}}$ to the top of $w$,
The relation between $V$ and $\mathcal{Q}\text{Aut}(T_{2,c})$
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- Apply $\sigma_{d_T}$ to the $n$-leaves and $b_{d_T}$ to the $p$-leaves.
The relation between $V$ and $\mathcal{QAut}(T_{2,c})$

- Apply $\sigma_{d_\tau}$ to the $n$-leaves and $b_{d_\tau}$ to the $p$-leaves.
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Lehnert’s conjecture revisited
Thompson’s group $V$ is the universal coCF group.
Work in progress on other subgroups of $V$

We are working on embedding other subgroups into $V$. Candidates we are looking at are surface groups:

\begin{align*}
\langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \ldots [a_n, b_n] \rangle \quad \text{(orientable)}
\end{align*}

\begin{align*}
\langle a_1, \ldots, a_n \mid a_2 \ldots a_n \rangle \quad \text{(non-orientable)}
\end{align*}

Recall:

- finite index subgroups of surface groups are still surface groups,
- there exist orientable double covers of non-orientable surfaces.

Theorem (Bleak-Salazar)

Let $H \leq V$. Any of its finite index overgroups is a subgroup of $V$. 
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Work in progress on other subgroups of $V$

We are attempting to build a surface group inside $V$. If one exists, then every other surface group will be in $V$.

Question: Do surface groups embed in $V$?

Surface groups are special cases of these Fuchsian groups:

$$\langle a_1, b_1, \ldots, a_n, b_n, c_1, \ldots, c_t, c^{-1}_1, \ldots, c^{-1}_t | a_1, b_1 \ldots a_n, b_n \rangle,$$

$n, s, t \geq 0$

Theorem (Fricke-Klein, Hoare-Karrass-Solitar)

Any finite index group of a Fuchsian group of the type above is a Fuchsian group of the same type.

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