On the influence of subgroups on the structure of finite groups

Izabela Agata Malinowska

Institute of Mathematics
University of Białystok, Poland

St Andrews, 3-11.08.2013
All groups considered here are finite.

A group $G$ is *Dedekind* if every subgroup of $G$ is normal in $G$. 

Theorem (R. Dedekind, 1896)

A group $G$ is Dedekind if and only if $G$ is abelian or $G$ is a direct product of the quaternion group $Q_8$ of order 8, an elementary abelian $2$-group and an abelian group of odd order.
All groups considered here are finite.

A group $G$ is **Dedekind** if every subgroup of $G$ is normal in $G$.

**Theorem (R. Dedekind, 1896)**

A group $G$ is Dedekind if and only if $G$ is abelian or $G$ is a direct product of the quaternion group $Q_8$ of order 8, an elementary abelian 2-group and an abelian group of odd order.
A subgroup $H$ of a group $G$ is \textit{permutable} in a group $G$ if $HK = KH$ whenever $K \leq G$. 

\begin{center}
\textbf{Theorem (O. Ore, 1939)}

If $H$ is a permutable subgroup of a group $G$, then $H$ is subnormal in $G$.

\textbf{Theorem (K. Iwasawa, 1941)}

Let $p$ be a prime. A $p$-group $G$ is an Iwasawa group if and only if $G$ is a Dedekind group, or $G$ contains an abelian normal subgroup $N$ such that $G/N$ is cyclic and $G = \langle x \rangle N$ for an element $x$ of $G$ and $x = a + p^s$ for all $a \in N$, where $s \geq 1$ and $s \geq 2$ if $p = 2$. 
\end{center}
A subgroup $H$ of a group $G$ is *permutable* in a group $G$ if $HK = KH$ whenever $K \leq G$.

**Theorem (O. Ore, 1939)**

If $H$ is a permutable subgroup of a group $G$, then $H$ is subnormal in $G$. 

---

**Theorem (K. Iwasawa, 1941)**

Let $p$ be a prime. A $p$-group $G$ is an Iwasawa group if and only if $G$ is a Dedekind group, or $G$ contains an abelian normal subgroup $N$ such that $G/N$ is cyclic and so $G = \langle x \rangle N$ for an element $x$ of $G$ and $x = a_1 + p^s$ for all $a \in N$, where $s \geq 1$ and $s \geq 2$ if $p = 2$. 

---

**Izabela Agata Malinowska**

On the influence of subgroups on the structure of groups
A subgroup \( H \) of a group \( G \) is \textit{permutable} in a group \( G \) if \( HK = KH \) whenever \( K \leq G \).

\begin{quote}
\textbf{Theorem (O. Ore, 1939)}

If \( H \) is a permutable subgroup of a group \( G \), then \( H \) is subnormal in \( G \).
\end{quote}

A group \( G \) is an \textit{Iwasawa group} if every subgroup of \( G \) is permutable in \( G \).
A subgroup $H$ of a group $G$ is *permutable* in a group $G$ if $HK = KH$ whenever $K \leq G$.

**Theorem (O. Ore, 1939)**

*If $H$ is a permutable subgroup of a group $G$, then $H$ is subnormal in $G$.*

A group $G$ is an *Iwasawa group* if every subgroup of $G$ is permutable in $G$.

**Theorem (K. Iwasawa, 1941)**

*Let $p$ be a prime. A $p$-group $G$ is an Iwasawa group if and only if $G$ is a Dedekind group, or $G$ contains an abelian normal subgroup $N$ such that $G/N$ is cyclic and so $G = \langle x \rangle N$ for an element $x$ of $G$ and $a^x = a^{1+p^s}$ for all $a \in N$, where $s \geq 1$ and $s \geq 2$ if $p = 2$.***
A subgroup of a group \( G \) is **s-permutable** in \( G \) if it permutes with all Sylow subgroups of \( G \).
A subgroup of a group $G$ is **s-permutable** in $G$ if it permutes with all Sylow subgroups of $G$.

**Theorem (O.H. Kegel, 1962)**

*If $H$ is an s-permutable subgroup of $G$, then $H$ is subnormal in $G$.***
The *nilpotent residual* of $G$ is the smallest normal subgroup of $G$ with nilpotent quotient.
Characterizations based on the normal structure

The *nilpotent residual* of $G$ is the smallest normal subgroup of $G$ with nilpotent quotient.

**Definition**

A group $G$ is a *T-group* if every subnormal subgroup of $G$ is normal in $G$. 
The *nilpotent residual* of $G$ is the smallest normal subgroup of $G$ with nilpotent quotient.

**Definition**

A group $G$ is a **$T$-group** if every subnormal subgroup of $G$ is normal in $G$.

**Examples of $T$-groups:**
- Dedekind groups = nilpotent $T$-groups;
- simple groups.
Characterizations based on the normal structure

Theorem (W. Gaschütz, 1957)

A group $G$ is a soluble $T$-group if and only if the following conditions are satisfied:

1. the nilpotent residual $L$ of $G$ is an abelian Hall subgroup of odd order;
2. $G$ acts by conjugation on $L$ as a group of power automorphisms, and
3. $G/L$ is a Dedekind group.

Definition

A group $G$ is said to be a PT-group when if $H$ is a permutable subgroup of $K$ and $K$ is a permutable subgroup of $G$, then $H$ is a permutable subgroup of $G$. 

Izabela Agata Malinowska

On the influence of subgroups on the structure of groups
Characterizations based on the normal structure

**Theorem (W. Gaschütz, 1957)**

A group $G$ is a soluble $T$-group if and only if the following conditions are satisfied:

1. the nilpotent residual $L$ of $G$ is an abelian Hall subgroup of odd order;
2. $G$ acts by conjugation on $L$ as a group of power automorphisms, and
3. $G/L$ is a Dedekind group.

**Definition**

A group $G$ is said to be a $PT$-group when if $H$ is a permutable subgroup of $K$ and $K$ is a permutable subgroup of $G$, then $H$ is a permutable subgroup of $G$. 
Examples of $PT$-groups:

- $T$-groups;
- Iwasawa groups = nilpotent $PT$-groups.
Characterizations based on the normal structure

Examples of $PT$-groups:

- $T$-groups;
- Iwasawa groups = nilpotent $PT$-groups.

The $PT$-groups are exactly the groups in which every subnormal subgroup is permutable.
Characterizations based on the normal structure

Examples of $PT$-groups:

- $T$-groups;
- Iwasawa groups = nilpotent $PT$-groups.

The $PT$-groups are exactly the groups in which every subnormal subgroup is permutable.

Theorem (G. Zacher, 1964)

A group $G$ is a soluble $PT$-group if and only if the following conditions are satisfied:

1. the nilpotent residual $L$ of $G$ is an abelian Hall subgroup of odd order;
2. $G$ acts by conjugation on $L$ as a group of power automorphisms, and
3. $G/L$ is an Iwasawa group.
Definition

A group $G$ is a \textit{PST-group} when if $H$ is an $s$-permutable subgroup of $K$ and $K$ is an $s$-permutable subgroup of $G$, then $H$ is an $s$-permutable subgroup of $G$. 

Examples of \textit{PST}-groups: nilpotent groups; \textit{PT}-groups. The \textit{PST}-groups are exactly the groups in which every subnormal subgroup is $s$-permutable.
Characterizations based on the normal structure

Definition

A group $G$ is a \textit{PST-group} when if $H$ is an $s$-permutable subgroup of $K$ and $K$ is an $s$-permutable subgroup of $G$, then $H$ is an $s$-permutable subgroup of $G$.

Examples of \textit{PST}-groups:

- nilpotent groups;
- \textit{PT}-groups.
Characterizations based on the normal structure

**Definition**

A group $G$ is a $PST$-group when if $H$ is an $s$-permutable subgroup of $K$ and $K$ is an $s$-permutable subgroup of $G$, then $H$ is an $s$-permutable subgroup of $G$.

**Examples of $PST$-groups:**

- nilpotent groups;
- $PT$-groups.

The $PST$-groups are exactly the groups in which every subnormal subgroup is $s$-permutable.
Theorem (R.K. Agrawal, 1975)

Let $G$ be a group with nilpotent residual $L$. The following statements are equivalent:

1. $L$ is an abelian Hall subgroup of odd order in which $G$ acts by conjugation as a group of power automorphisms;
2. $G$ is a soluble PST-group.

Corollary

- $G$ is a soluble PT-group if and only if $G$ is a soluble PST-group whose Sylow subgroups are Iwasawa groups;
- $G$ is a soluble T-group if and only if $G$ is a soluble PST-group whose Sylow subgroups are Dedekind groups.
Theorem (R.K. Agrawal, 1975)

Let $G$ be a group with nilpotent residual $L$. The following statements are equivalent:

1. $L$ is an abelian Hall subgroup of odd order in which $G$ acts by conjugation as a group of power automorphisms;
2. $G$ is a soluble PST-group.

Corollary

Let $G$ be a group.

1. $G$ is a soluble PT-group if and only if $G$ is a soluble PST-group whose Sylow subgroups are Iwasawa groups;
2. $G$ is a soluble $T$-group if and only if $G$ is a soluble PST-group whose Sylow subgroups are Dedekind groups.
In the soluble universe:

\[
\begin{array}{cccc}
T & \subseteq & PT & \subseteq \\
\subseteq & \subseteq & \subseteq & \subseteq \\
\text{Dedekind} & \subseteq & \text{Iwasawa} & \subseteq \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{supersoluble} & \subseteq & \text{nilpotent} & \subseteq \\
\subseteq & \subseteq & \subseteq & \subseteq \\
\end{array}
\]

Izabela Agata Malinowska

On the influence of subgroups on the structure of groups
If $H$ is a subgroup of a group $G$, we denote by $H^G$ the normal closure of $H$ in $G$, that is, the smallest normal subgroup of $G$ containing $H$.

**Theorem (Y. Li, 2006)**

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble $T$-group;
2. $N_G(H) \cap H^G = H$ for all subgroups $H$ of $G$;
3. $N_G(H) \cap H^G = H$ for all $p$-subgroups $H$ of $G$ and every prime $p$. 

Izabela Agata Malinowska

On the influence of subgroups on the structure of groups
Theorem (O.H. Kegel, 1962)

If $H_1$ and $H_2$ are two s-permutable subgroups of the group $G$, then $H_1 \cap H_2$ is an s-permutable subgroup of $G$. Consequently, the set of all s-permutable subgroups is a sublattice of the subnormal subgroup lattice.
Characterizations based on subgroup embedding properties

Theorem (O.H. Kegel, 1962)

If $H_1$ and $H_2$ are two s-permutable subgroups of the group $G$, then $H_1 \cap H_2$ is an s-permutable subgroup of $G$. Consequently, the set of all s-permutable subgroups is a sublattice of the subnormal subgroup lattice.

Definition

Let $H$ be a subgroup of a group $G$.

1. The permutable closure $A_G(H)$ of $H$ in $G$ is the intersection of all permutable subgroups of $G$ containing $H$.

2. The s-permutable closure $B_G(H)$ of $H$ in $G$ is the intersection of all s-permutable subgroups of $G$ containing $H$. 
Example

Assume that $p$ is an odd prime,

$$A = \langle a, b \mid a^{p^2} = b^p = 1, a^b = a^{1+p} \rangle$$

is an extraspecial group of order $p^3$ and exponent $p^2$ and $Z = \langle z \rangle$ is a cyclic group of order $p^2$. Consider $G = A \times Z$. Then $A \triangleleft G$ and $B = \langle b \rangle \times \langle z \rangle$ is permutable in $G$ since $\langle b \rangle$ is permutable in $A$. But $A \cap B = \langle b \rangle$ is not permutable in $G$, since $\langle b \rangle$ does not permute with $\langle az \rangle$. For a subgroup $H = \langle b \rangle$, the permutable closure $A_G(H) = H$ is not permutable in $G$. 

Izabela Agata Malinowska

On the influence of subgroups on the structure of groups
Theorem (A. Ballester-Bolinches, R. Esteban-Romero, Y. Li, 2010)

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PT-group;
2. $N_G(H) \cap A_G(H) = H$ for every subgroup $H$ of $G$;

Izabela Agata Malinowska
On the influence of subgroups on the structure of groups
Characterizations based on subgroup embedding properties

**Theorem (A. Ballester-Bolinches, R. Esteban-Romero, Y. Li, 2010)**

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PT-group;
2. $N_G(H) \cap A_G(H) = H$ for every subgroup $H$ of $G$;

**Theorem (A. Ballester-Bolinches, R. Esteban-Romero, Y. Li, 2010)**

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PST-group;
2. $N_G(H) \cap B_G(H) = H$ for every subgroup $H$ of $G$;
A subgroup $H$ of a group $G$ is an \textit{NR-subgroup} of $G$ (Normal Restriction) if, whenever $K \trianglelefteq H$, $K^G \cap H = K$.

Example

Let $G = A_5$, the alternating group of degree 5. Then every 5-subgroup of $G$ is an NR-subgroup of $G$, a PR-subgroup of $G$ and an sPR-subgroup of $G$. Let $H = \langle (12345) \rangle$. Hence $|N^G(H)| = 10$ and $H^G \cap N^G(H) = N^G(H) \cap B^G(H) = N^G(H) \neq H$. 

Izabela Agata Malinowska
On the influence of subgroups on the structure of groups
A subgroup \( H \) of a group \( G \) is an **NR-subgroup** of \( G \) (Normal Restriction) if, whenever \( K \trianglelefteq H \), \( K^G \cap H = K \).

A subgroup \( H \) of a group \( G \) is said to be a **PR-subgroup** of \( G \) (Permutable Restriction) if, whenever \( K \trianglelefteq H \), \( A_G(K) \cap H = K \).
A subgroup $H$ of a group $G$ is an **NR-subgroup** of $G$ (Normal Restriction) if, whenever $K \trianglelefteq H$, $K^G \cap H = K$.

A subgroup $H$ of a group $G$ is said to be a **PR-subgroup** of $G$ (Permutable Restriction) if, whenever $K \trianglelefteq H$, $A_G(K) \cap H = K$.

A subgroup $H$ of a group $G$ is said to be an **sPR-subgroup** of $G$ (s-Permutable Restriction) if, whenever $K \trianglelefteq H$, $B_G(K) \cap H = K$. 

---

**Example**

Let $G = A_5$, the alternating group of degree 5. Then every 5-subgroup of $G$ is an NR-subgroup of $G$, a PR-subgroup of $G$ and an sPR-subgroup of $G$. Let $H = \langle (12345) \rangle$. Hence $|N_G(H)| = 10$ and $H^G \cap N_G(H) = N_G(H) \cap A_G(H) = N_G(H) \neq H$. 

---

Izabela Agata Malinowska

On the influence of subgroups on the structure of groups
A subgroup $H$ of a group $G$ is an **NR-subgroup** of $G$ (Normal Restriction) if, whenever $K \trianglelefteq H$, $K^G \cap H = K$.

A subgroup $H$ of a group $G$ is said to be a **PR-subgroup** of $G$ (Permutable Restriction) if, whenever $K \trianglelefteq H$, $A_G(K) \cap H = K$.

A subgroup $H$ of a group $G$ is said to be an **sPR-subgroup** of $G$ (s-Permutable Restriction) if, whenever $K \trianglelefteq H$, $B_G(K) \cap H = K$.

### Example

Let $G = A_5$, the alternating group of degree 5. Then every 5-subgroup of $G$ is an NR-subgroup of $G$, a PR-subgroup of $G$ and an sPR-subgroup of $G$. Let $H = \langle (12345) \rangle$. Hence $|N_G(H)| = 10$ and $H^G \cap N_G(H) = N_G(H) \cap A_G(H) = N_G(H) \cap B_G(H) = N_G(H) \neq H$. 

Izabela Agata Malinowska 

On the influence of subgroups on the structure of groups
Example

Let $G$ be the semidirect product of a quaternion group $P$ of order 8 with a cyclic group $Q$ of order 3, which induces an automorphism permuting cyclically the three maximal subgroups of the quaternion group. Then every 3-subgroup of $G$ is an $NR$-subgroup of $G$, a $PR$-subgroup of $G$ and an $sPR$-subgroup of $G$. But

$$Q^G \cap N_G(Q) = A_G(Q) \cap N_G(Q) = B_G(Q) \cap N_G(Q) = G \cap QP' = QP' \neq Q.$$
Theorem (I.A.M., 2012)

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble $T$-group;
2. each subgroup of $G$ is an NR-subgroup of $G$;
3. for each prime $p \in \pi(G)$, each $p$-subgroup of $G$ is an NR-subgroup of $G$. 
A subgroup $H$ of a group $G$ is \textit{normal sensitive in $G$} if the following holds:
\[
\{ N \mid N \text{ is normal in } H \} = \{ H \cap W \mid W \text{ is normal in } G \}.
\]
A subgroup $H$ of a group $G$ is *normal sensitive in $G$* if the following holds:

$$\{ N \mid N \text{ is normal in } H \} = \{ H \cap W \mid W \text{ is normal in } G \}.$$ 

**Theorem (S. Bauman, 1974)**

*Every subgroup of a group $G$ is normal sensitive in $G$ if and only if $G$ is a soluble $T$-group.*
A subgroup $H$ of a group $G$ is normal sensitive in $G$ if the following holds:

$$\{ N \mid N \text{ is normal in } H \} = \{ H \cap W \mid W \text{ is normal in } G \}.$$

**Theorem (S. Bauman, 1974)**

Every subgroup of a group $G$ is normal sensitive in $G$ if and only if $G$ is a soluble $T$-group.

**Corollary (I.A.M., 2012)**

A group $G$ is a soluble $T$-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is normal sensitive in $G$. 
Theorem (I.A.M., 2012)

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PT-group;
2. each subgroup of $G$ is a PR-subgroup of $G$;
3. for each prime $p \in \pi(G)$, each $p$-subgroup of $G$ is a PR-subgroup of $G$. 

A subgroup $H$ of a group $G$ is permutable sensitive in $G$ if the following holds:

$$\left\{ N \mid N \text{ is permutable in } H \right\} = \left\{ H \cap W \mid W \text{ is permutable in } G \right\}.$$

Theorem (J.C. Beidleman, M.F. Rangland, 2007)

Every subgroup of a group $G$ is permutable sensitive in $G$ if and only if $G$ is a soluble PT-group.
Theorem (I.A.M., 2012)

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PT-group;
2. each subgroup of $G$ is a PR-subgroup of $G$;
3. for each prime $p \in \pi(G)$, each $p$-subgroup of $G$ is a PR-subgroup of $G$.

A subgroup $H$ of a group $G$ is permutable sensitive in $G$ if the following holds:

$$\{N \mid N \text{ is permutable in } H\} = \{H \cap W \mid W \text{ is permutable in } G\}.$$
Theorem (I.A.M., 2012)

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PT-group;
2. each subgroup of $G$ is a PR-subgroup of $G$;
3. for each prime $p \in \pi(G)$, each $p$-subgroup of $G$ is a PR-subgroup of $G$.

A subgroup $H$ of a group $G$ is *permutable sensitive in $G$* if the following holds:

$$\{N \mid N \text{ is permutable in } H\} = \{H \cap W \mid W \text{ is permutable in } G\}.$$

Theorem (J.C. Beidleman, M.F. Rangland, 2007)

Every subgroup of a group $G$ is permutable sensitive in $G$ if and only if $G$ is a soluble PT-group.
Corollary (I.A.M., 2012)

A group $G$ is a soluble $PT$-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is permutable sensitive in $G$.
Corollary (I.A.M., 2012)

A group $G$ is a soluble PT-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is permutable sensitive in $G$.

Example

Let $P = \langle a, b \mid a^3 = b^{3^2} = 1, b^a = b^{4^2} \rangle$ be a metacyclic group of order $3^3$ and exponent $3^2$. Let $x$ be the automorphism of $P$ of order 2 given by $a^x = a$, $b^x = b^{-1}$. Let $H = P \rtimes \langle x \rangle$ be the corresponding semidirect product and let $G = H \rtimes C$, where $C = \langle c \rangle$ is cyclic of order 3. Then a subgroup $\langle a, bc \rangle$ is a $PR$-subgroup of $G$. But it is not permutable sensitive in $G$. 
Theorem (I.A.M., 2012)

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PST-group;
2. each subgroup of $G$ is an sPR-subgroup of $G$;
3. for each prime $p \in \pi(G)$, every $p$-subgroup of $G$ is an sPR-subgroup of $G$. 

A subgroup $H$ of a group $G$ is s-permutable sensitive in $G$ if the following holds:

\[ \{N \mid N \text{ is s-permutable in } H\} = \{H \cap W \mid W \text{ is s-permutable in } G\}. \]
Theorem (I.A.M., 2012)

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PST-group;
2. each subgroup of $G$ is an $sPR$-subgroup of $G$;
3. for each prime $p \in \pi(G)$, every $p$-subgroup of $G$ is an $sPR$-subgroup of $G$.

A subgroup $H$ of a group $G$ is $s$-permutable sensitive in $G$ if the following holds:
\[ \{ N \mid N \text{ is } s\text{-permutable in } H \} = \{ H \cap W \mid W \text{ is } s\text{-permutable in } G \}. \]
Characterizations based on subgroup embedding properties

**Theorem (I.A.M., 2012)**

Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PST-group;
2. each subgroup of $G$ is an sPR-subgroup of $G$;
3. for each prime $p \in \pi(G)$, every $p$-subgroup of $G$ is an sPR-subgroup of $G$.

A subgroup $H$ of a group $G$ is *s-permutable sensitive in $G$* if the following holds:

$$\{ N \mid N \text{ is } s\text{-permutable in } H \} = \{ H \cap W \mid W \text{ is } s\text{-permutable in } G \}.$$

**Theorem (J.C. Beidleman, M.F. Rangland, 2007)**

Every subgroup of a group $G$ is *s-permutable sensitive in $G$* if and only if $G$ is a soluble PST-group.
Corollary (I.A.M., 2012)

A group $G$ is a soluble PST-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is s-permutable sensitive in $G$. 

Example

Let $G$ be the direct product of a symmetric group of degree 4 and a cyclic group of order 2. Let $H = \langle (1, 2), (1, 3)(2, 4)(5, 6), (1, 2)(3, 4) \rangle$ (here $(5, 6)$ generates the cyclic subgroup of order 2). Then $H$ is an sPR-subgroup of $G$, but it is not s-permutable sensitive in $G$. 

Izabela Agata Malinowska

On the influence of subgroups on the structure of groups
Corollary (I.A.M., 2012)

A group $G$ is a soluble PST-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is $s$-permutable sensitive in $G$.

Example

Let $G$ be the direct product of a symmetric group of degree 4 and a cyclic group of order 2. Let $H = \langle (1, 2), (1, 3)(2, 4)(5, 6), (1, 2)(3, 4) \rangle$ (here $(5, 6)$ generates the cyclic subgroup of order 2). Then $H$ is an $sPR$-subgroup of $G$, but it is not $s$-permutable sensitive in $G$. 
A subgroup $H$ of $G$ is an $\mathcal{H}$-subgroup of $G$ if $N_G(H) \cap H^g \leq H$ for all $g \in G$. 
Local characterizations

A subgroup $H$ of $G$ is an $\mathcal{H}$-subgroup of $G$ if $N_G(H) \cap H^g \leq H$ for all $g \in G$.

Let $p$ be a prime. A group $G$ satisfies:

- **the property $NR_p$** if a Sylow $p$-subgroup of $G$ is an $NR$-subgroup of $G$;
- **the property $H_p$** if every maximal subgroup of a Sylow $p$-subgroup of $G$ is an $\mathcal{H}$-subgroup of $G$. 

Theorem (I.A.M., 2012)

Let $G$ be a group. The following conditions are equivalent:

1. $G$ is a soluble PST-group;
2. every subgroup of $G$ satisfies $NR_p$ for every prime $p \in \pi(G)$;
3. every subgroup of $G$ satisfies $H_p$ for every prime $p \in \pi(G)$.
A subgroup $H$ of $G$ is an $\mathcal{H}$-subgroup of $G$ if $N_G(H) \cap H^g \leq H$ for all $g \in G$.

Let $p$ be a prime. A group $G$ satisfies:

- the property $NR_p$ if a Sylow $p$-subgroup of $G$ is an $NR$-subgroup of $G$;
- the property $\mathcal{H}_p$ if every maximal subgroup of a Sylow $p$-subgroup of $G$ is an $\mathcal{H}$-subgroup of $G$.

**Theorem (I.A.M., 2012)**

Let $G$ be a group. The following conditions are equivalent:

1. $G$ is a soluble PST-group;
2. every subgroup of $G$ satisfies $NR_p$ for every prime $p \in \pi(G)$;
3. every subgroup of $G$ satisfies $\mathcal{H}_p$ for every prime $p \in \pi(G)$. 

Izabela Agata Malinowska  
On the influence of subgroups on the structure of groups
Theorem (I.A.M., 2012)

Let $G$ be a group all of whose second maximal subgroups of even order are soluble PST-groups. Then $G$ is either a soluble group or one of the following groups:

1. $\text{PSL}(2, 2^f)$, where $f$ is a prime such that $2^f - 1$ is a prime;
2. $\text{PSL}(2, p)$, where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;
3. $\text{PSL}(2, 3^f)$, where $f$ is an odd prime and $3^f \equiv 3 \pmod{8}$;
4. $\text{SL}(2, 3^f)$, where $f$ is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is a prime;
5. $\text{SL}(2, p)$, where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;
Theorem (I.A.M., 2012)

Let $G$ be a group all of whose second maximal subgroups are soluble PST-groups. Then $G$ is either a soluble group or one of the following groups:

1. $\text{PSL}(2, 2^f)$, where $f$ is a prime such that $2^f - 1$ is a prime;
2. $\text{PSL}(2, p)$, where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$;
3. $\text{PSL}(2, 3^f)$, where $f$ is an odd prime, $3^f \equiv 3 \pmod{8}$ and $(3^f - 1)/2$ is a prime;
4. $\text{SL}(2, p)$, where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod{5}$ and $p \equiv 3$ or $5 \pmod{8}$. 
Let $p$ be a prime number.
Let $p$ be a prime number.

a group $G$ satisfies the property $C_p$ if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in its normalizer $N_G(P)$,
Let $p$ be a prime number.

a group $G$ satisfies the property $C_p$ if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in its normalizer $N_G(P),$

$G$ satisfies $X_p$ if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is permutable in $N_G(P),$

Theorem (A. Ballester-Bolinches, R. Esteban-Romero, 2002)

A group $G$ satisfies $X_p$ (respectively, $C_p$) if and only if $G$ satisfies $Y_p$ and the Sylow $p$-subgroups of $G$ are Iwasawa (respectively, Dedekind).
Let $p$ be a prime number.

A group $G$ satisfies the property $C_p$ if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in its normalizer $N_G(P)$.

$G$ satisfies $X_p$ if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is permutable in $N_G(P)$.

$G$ satisfies $Y_p$ if, whenever $H$ and $K$ are $p$-subgroups of $G$ with $H \leq K$, $H$ is $s$-permutable in $N_G(K)$. 

Theorem (A. Ballester-Bolinches, R. Esteban-Rormero, 2002)

A group $G$ satisfies $X_p$ (respectively, $C_p$) if and only if $G$ satisfies $Y_p$ and the Sylow $p$-subgroups of $G$ are Iwasawa (respectively, Dedekind).
Local characterizations

Let $p$ be a prime number.

A group $G$ satisfies the property $\mathcal{C}_p$ if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in its normalizer $N_G(P)$,

$G$ satisfies $\mathcal{X}_p$ if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is permutuble in $N_G(P)$,

$G$ satisfies $\mathcal{Y}_p$ if, whenever $H$ and $K$ are $p$-subgroups of $G$ with $H \leq K$, $H$ is $s$-permutable in $N_G(K)$.

**Theorem (A. Ballester-Bolinches, R. Esteban-Romero, 2002)**

A group $G$ satisfies $\mathcal{X}_p$ (respectively, $\mathcal{C}_p$) if and only if $G$ satisfies $\mathcal{Y}_p$ and the Sylow $p$-subgroups of $G$ are Iwasawa (respectively, Dedekind).
Let $p$ be a prime.

\[
\begin{align*}
\mathcal{C}_p & \subsetneq \mathcal{X}_p & \subsetneq \mathcal{Y}_p \\
\text{Dedekind } p\text{-groups} & \subsetneq \text{Iwasawa } p\text{-groups} & \subsetneq p\text{-groups}
\end{align*}
\]
Theorem (D.J.S. Robinson, 1968)

A group $G$ is a soluble $T$-group if and only if $G$ satisfies $C_p$ for all $p \in \pi(G)$.

Theorem (J.C. Beidleman, B. Brewster, D.J.S. Robinson, 1999)

A group $G$ is a soluble PT-group if and only if $G$ satisfies $X_p$ for all $p \in \pi(G)$.

Theorem (A. Ballester-Bolinches, R. Esteban-Romero, 2002)

A group $G$ a soluble PST-group if and only if $G$ satisfies $Y_p$ for all $p \in \pi(G)$. 
Theorem (D.J.S. Robinson, 1968)

A group $G$ is a soluble $T$-group if and only if $G$ satisfies $C_p$ for all $p \in \pi(G)$.

Theorem (J.C. Beidleman, B. Brewster, D.J.S. Robinson, 1999)

A group $G$ is a soluble $PT$-group if and only if $G$ satisfies $X_p$ for all $p \in \pi(G)$.
Local characterizations

**Theorem (D.J.S. Robinson, 1968)**

A group $G$ is a soluble $T$-group if and only if $G$ satisfies $C_p$ for all $p \in \pi(G)$.

**Theorem (J.C. Beidleman, B. Brewster, D.J.S. Robinson, 1999)**

A group $G$ is a soluble $PT$-group if and only if $G$ satisfies $\chi_p$ for all $p \in \pi(G)$.

**Theorem (A. Ballester-Bolinches, R. Esteban-Romero, 2002)**

A group $G$ is a soluble $PST$-group if and only if $G$ satisfies $\gamma_p$ for all $p \in \pi(G)$.
Local characterizations

A subgroup $H$ of a group $G$ is said to be \textit{pronormal in $G$} if for every $g \in G$, $H$ and $H^g$ are conjugate in their join $\langle H, H^g \rangle$.

A group $G$ satisfies \textit{the property $H_p$} if every normal subgroup of a Sylow $p$-subgroup of $G$ is pronormal in $G$. 

Theorem

Let $G$ be a group and let $p$ be a prime. Then:

2. (J.C. Beidleman, B. Brewster, D.J.S. Robinson, 1999) $G$ satisfies $X_p$ if and only if $G$ satisfies $H_p$ and $G$ has Iwasawa Sylow $p$-subgroups.

Proposition

Let $G$ be a group and let $p$ be a prime.

1. If $G$ satisfies $NR_p$, then $G$ satisfies the property $H_p$.
2. If $G$ satisfies $NR_p$, then $G$ satisfies the property $H_p$.

Izabela Agata Malinowska

On the influence of subgroups on the structure of groups
Local characterizations

A subgroup $H$ of a group $G$ is said to be **pronormal in $G$** if for every $g \in G$, $H$ and $H^g$ are conjugate in their join $\langle H, H^g \rangle$.

A group $G$ satisfies the property $\mathbf{H}_p$ if every normal subgroup of a Sylow $p$-subgroup of $G$ is pronormal in $G$.

**Theorem**

Let $G$ be a group and let $p$ be a prime. Then:

1. (A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, 2010) $G$ satisfies $\mathbf{Y}_p$ if and only if every subgroup of $G$ satisfies $\mathbf{H}_p$;
2. (J.C. Beidleman, B. Brewster, D.J.S. Robinson, 1999) $G$ satisfies $\mathbf{X}_p$ if and only if $G$ satisfies $\mathbf{H}_p$ and $G$ has Iwasawa Sylow $p$-subgroups.
Local characterizations

A subgroup $H$ of a group $G$ is said to be *pronormal in $G$* if for every $g \in G$, $H$ and $H^g$ are conjugate in their join $\langle H, H^g \rangle$.

A group $G$ satisfies *the property $H_p$* if every normal subgroup of a Sylow $p$-subgroup of $G$ is pronormal in $G$.

**Theorem**

Let $G$ be a group and let $p$ be a prime. Then:

1. (A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, 2010) $G$ satisfies $\mathcal{Y}_p$ if and only if every subgroup of $G$ satisfies $H_p$;
2. (J.C. Beidleman, B. Brewster, D.J.S. Robinson, 1999) $G$ satisfies $\mathcal{X}_p$ if and only if $G$ satisfies $H_p$ and $G$ has Iwasawa Sylow $p$-subgroups.

**Proposition**

Let $G$ be a group and let $p$ be a prime.

1. If $G$ satisfies $NR_p$, then $G$ satisfies the property $H_p$.
2. If $G$ satisfies $NR_p$, then $G$ satisfies the property $H_p$. 

Izabela Agata Malinowska  
On the influence of subgroups on the structure of groups
Example
Let $p$ be an odd prime and let $A = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c \rangle$ be an extraspecial group of order $p^3$ and exponent $p$. Let $B = \langle x \rangle$ be a cyclic group of order $p$ and $P = A \times B$. Let $y$ be an automorphism of $P$ of order 2 given by $a^y = a^{-1}$, $b^y = b^{-1}$, $x^y = x^{-1}$. Let $G = P \rtimes \langle y \rangle$ be the corresponding semidirect product. Then every maximal subgroup of $P$ is normal in $G$, so is an $\mathcal{H}$-subgroup of $G$. Hence $G$ satisfies $\mathcal{H}_p$. But $H = \langle xc \rangle$ is normal in $P$, $\langle xc \rangle^y = \langle x^{-1}c \rangle$, $\langle xc \rangle$ and $\langle x^{-1}c \rangle$ are not conjugate in $\langle x, c \rangle$. Therefore $G$ satisfies neither $\mathcal{H}_p$ nor $\mathcal{N}\mathcal{R}_p$. 
Theorem (I.A.M., 2012)

Let $p$ be a prime and let $G$ be a $p$-soluble group. Then every subgroup of $G$ satisfies $H_p$ if and only if every subgroup of $G$ satisfies $NR_p$. 

Izabela Agata Malinowska

On the influence of subgroups on the structure of groups
Theorem (I.A.M., 2012)

Let \( p \) be a prime and let \( G \) be a \( p \)-soluble group. Then every subgroup of \( G \) satisfies \( H_p \) if and only if every subgroup of \( G \) satisfies \( NR_p \).

Theorem (I.A.M., 2012)

Let \( p \) be a prime and let \( G \) be a \( p \)-soluble group. Then \( G \) satisfies \( H_p \) if and only if \( G \) satisfies \( NR_p \).
Theorem (I.A.M., 2012)

Let $p$ be a prime and let $G$ be a $p$-soluble group. Then every subgroup of $G$ satisfies $H_p$ if and only if every subgroup of $G$ satisfies $NR_p$.

Theorem (I.A.M., 2012)

Let $p$ be a prime and let $G$ be a $p$-soluble group. Then $G$ satisfies $H_p$ if and only if $G$ satisfies $NR_p$.

Theorem (I.A.M., 2012)

Let $p$ be a prime and let $G$ be a $p$-soluble group. Then every subgroup of $G$ satisfies $NR_p$ if and only if every subgroup of $G$ satisfies $H_p$. 
Local characterizations

Example

Let $G = PSL(2, 53)$. Since a Sylow 3-subgroup of $G$ is cyclic of order $3^3$, $G$ and its subgroups satisfy $H_3$ and $H_3$, but $G$ does not satisfy $NR_3$. 
Local characterizations

Example

Let $G = PSL(2, 53)$. Since a Sylow 3-subgroup of $G$ is cyclic of order $3^3$, $G$ and its subgroups satisfy $H_3$ and $\mathcal{H}_3$, but $G$ does not satisfy $NR_3$.

Question

Let $G$ be a non-$p$-soluble group. Is it true that every subgroup of $G$ satisfies $H_p$ if and only if every subgroup of $G$ satisfies $\mathcal{H}_p$?
Local characterizations

Example

Let $G = PSL(2, 53)$. Since a Sylow 3-subgroup of $G$ is cyclic of order $3^3$, $G$ and its subgroups satisfy $H_3$ and $H_3$, but $G$ does not satisfy $NR_3$.

Question

Let $G$ be a non-$p$-soluble group. Is it true that every subgroup of $G$ satisfies $H_p$ if and only if every subgroup of $G$ satisfies $H_p$?

Question

Assume that $G$ is a $p$-soluble group and $G$ has Iwasawa Sylow $p$-subgroups. Is it true that $G$ satisfies $NR_p$ if and only if $G$ satisfies $H_p$?
Corollary (I.A.M., 2012)

Let $p$ be a prime and let $G$ be a $p$-soluble group. Then:

1. $G$ satisfies $\gamma_p$ if and only if every subgroup of $G$ satisfies $NR_p$.
2. $G$ satisfies $\chi_p$ if and only if $G$ satisfies $NR_p$ and $G$ has Iwasawa Sylow $p$-subgroups.
**Corollary (I.A.M., 2012)**

Let $p$ be a prime and let $G$ be a $p$-soluble group. Then:

1. $G$ satisfies $\gamma_p$ if and only if every subgroup of $G$ satisfies $NR_p$.
2. $G$ satisfies $\chi_p$ if and only if $G$ satisfies $NR_p$ and $G$ has Iwasawa Sylow $p$-subgroups.

**Theorem (I.A.M., 2012)**

Let $G$ be a group. The following conditions are equivalent:

1. $G$ is a soluble PT-group;
2. $G$ satisfies $NR_p$ and $G$ has Iwasawa Sylow $p$-subgroups for all $p \in \pi(G)$. 

(2) I.A. Malinowska, *Finite groups with NR-subgroups or their generalizations*, J. Group Theory **15** (2012), 687–707.


and references in them.
Thank you