Commuting probability and commutator relations

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Groups St Andrews 2013
Commuting probability

Let $G$ be a *finite* group. The probability that a randomly chosen pair of elements of $G$ commute is called the **commuting probability** of $G$.

$$cp(G) = \frac{|\{(x, y) \in G \times G \mid [x, y] = 1\}|}{|G|^2}$$

- $cp(G) = k(G)/|G|$  
  Erdös, Turán 1968
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Outlook

- Global  Analyse the image of $\text{cp}$.
- Local    Study the impact $\text{cp}(G)$ has on the structure of $G$. 
Commuting probability globally
As a function on groups of order $\leq 256$
Commuting probability globally
As a function on groups of order $\leq 256 +$
Conjecture (Joseph 1977)

1. The limit points of \( \text{im cp} \) are rational.
2. If \( \ell \) is a limit point of \( \text{im cp} \), then there is an \( \varepsilon > 0 \) such that \( \text{im cp} \cap (\ell - \varepsilon, \ell) = \emptyset \).
3. \( \text{im cp} \cup \{0\} \) is a closed subset of \([0, 1]\).

• 1. and 2. hold for limit points \( \geq 2/9 \).  
  Hegarty 2012
Commuting probability locally

As a measure of being abelian

- If $\text{cp}(G) > 5/8$, then $G$ is abelian.  
  Gustafson 1973
- If $\text{cp}(G) > 1/2$, then $G$ is nilpotent.  
  Lescot 1988
- $\text{cp}(G) < |G : \text{Fit}(G)|^{-1/2}$  
  Guralnick, Robinson 2006
Commuting probability locally

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General principle

Bounding $\text{cp}(G)$ away from zero ensures abelian-like properties of $G$. 
The exterior square $G \wedge G$ of $G$ is the group generated by the symbols $x \wedge y$ for all $x, y \in G$, subject to universal commutator relations:

$$x \wedge x = 1, \quad xy \wedge z = (x^y \wedge z^y)(y \wedge z), \quad x \wedge yz = (x \wedge z)(x^z \wedge y^z).$$
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Setting up the terrain

The **exterior square** $G \wedge G$ of $G$ is the group generated by the symbols $x \wedge y$ for all $x, y \in G$, subject to *universal commutator relations*:

$$x \wedge x = 1, \quad xy \wedge z = (x^y \wedge z^y)(y \wedge z), \quad x \wedge yz = (x \wedge z)(x^z \wedge y^z).$$

The **curly exterior square** $G \circlearrowleft G$ of $G$ is the group generated by the symbols $x \circlearrowleft y$ for all $x, y \in G$, subject to *universal commutator relations, but without redundancies*, i.e.

$$G \circlearrowleft G = \frac{G \wedge G}{\langle x \wedge y \mid [x, y] = 1 \rangle}.$$
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Bogomolov multiplier

There is a natural commutator homomorphism $\kappa : G \ltimes G \to [G, G]$.

The kernel of $\kappa$ consists of non-universal commutator relations. This is the **Bogomolov multiplier** of the group $G$, denoted by $B_0(G)$.
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The group $B_0(G)$ is isomorphic to the unramified Brauer group of $G$, an obstruction to Noether’s problem of stable rationality of fixed fields.

- $\text{Br}_{nr}(\mathbb{C}(G)/\mathbb{C})$ embeds into $H^2(G, \mathbb{Q}/\mathbb{Z})$. Bogomolov 1987
- The image of the embedding is $B_0(G)^*$. Moravec 2012
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Bogomolov multiplier: examples

$B_0 = 0$

- Abelian-by-cyclic groups
- Finite simple groups
- Frobenius groups with abelian kernel
- $p$-groups of order $\leq p^4$
- Most groups of order $p^5$
- Unitriangular $p$-groups

$B_0 \neq 0$

- Smallest possible order is 64.
- $\langle a, b, c, d \mid [a, b] = [c, d], \exp 4, \cl 2 \rangle$

Bogomolov 1988
Kunyavskiǐ 2010
Moravčik 2012
Bogomolov 1988
Moravčik 2012
Chu, Hu, Kang, Kunyavskiǐ 2010
Commuting probability locally
The general principle universally

Theorem
If \( \text{cp}(G) > 1/4 \), then \( B_0(G) = 0 \).
Commuting probability locally

The general principle universally

**Theorem**

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**Outline of proof**

Assume that \( G \) is a group of the smallest possible order satisfying \( cp(G) > 1/4 \) and \( B_0(G) \neq 0 \). By standard arguments, \( G \) is a stem \( p \)-group.

Proper subgroups and quotients of \( G \) have a larger commuting probability than \( G \), so: \( B_0(G) \neq 0 \), but all proper subgroups and quotients of \( G \) have a trivial Bogomolov multiplier. Groups with the latter property are called \( B_0 \)-minimal.
A $B_0$-minimal group enjoys the following properties.

- Is a capable $p$-group with an abelian Frattini subgroup.
- Is of Frattini rank $\leq 4$.
- For stem groups, the exponent is bounded by a function of class alone.
$B_0$-minimal groups

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- Is a capable $p$-group with an abelian Frattini subgroup.
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- Given the nilpotency class, there are only finitely many isoclinism families containing a $B_0$-minimal group of this class.
- Classification of $B_0$-minimal groups of class 2, hence of class 2 groups of orders $p^7$ with non-trivial Bogomolov multipliers.
- Construction of a sequence of 2-groups with non-trivial Bogomolov multipliers and arbitrarily large nilpotency class.
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Considering the structure of \( B_0 \)-minimal groups of coclass 3, use the class equation to obtain bounds on the sizes of conjugacy classes of a suitably chosen generating set of \( G \). This restricts the nilpotency class of \( G \). Finish with the help of NQ.