Representations Arising from an Action on D-neighborhoods of Cayley Graphs

Justin Hughes
Colorado State University

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Outline

Cayley Graphs and Neighborhood Complexes

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- Distance set $D \in \{\{a_1, a_2, ..., a_n\} | a_i \in \mathbb{Z}_{\geq 0}\}$
- $N_D(x) = \{x_i | x_i \in V$ such that $d(x, x_i) \in D\}$
- Neighborhood complex: Simplicial complex with $N_D(x)\ \forall x \in G$ as faces.

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- $D = \{0, 1\}$

Chain Complexes and Homology

Let Δ denote a simplicial complex and $F_i(\Delta)$ denote the faces of dimension i.

Reduced Chain Complex:

$$0 \longrightarrow \mathbb{K}^{|F_n(\Delta)|} \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} \mathbb{K}^{|F_1(\Delta)|} \xrightarrow{\partial_1} \mathbb{K}^{|F_0(\Delta)|} \xrightarrow{\partial_0} \mathbb{K} \longrightarrow 0$$

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- $\partial_i(e_\alpha) = \sum_{j\in\alpha} \mathrm{sgn}(j,\alpha) e_{\alpha \smallsetminus j}$
- $\widetilde{H_i}(\Delta; \mathbb{K})$, is the vector space $\ker(\partial_i)/\mathrm{im}(\partial_{i+1})$

Neighborhoods Sharing an Edge

Theorem

Let $S = \{g_1, ..., g_n\}$ be an irredundant generating set for a group G. Denote the Cayley graph for G and S by $Cayley(G, S)$. Suppose there is an 1-simplex $\varepsilon \in N_{\{0,1\}}(x) \cap N_{\{0,1\}}(y)$ where $N_{\{0,1\}}(x)$ and $N_{\{0,1\}}(y)$ are distinct neighborhoods. Then either
$\varepsilon = F_1(x, g_i \cdot x) = F_1(y, g_i \cdot y)$ and $|g_i| = 2$ or
$\varepsilon = F_1(g_i \cdot x, g_j \cdot x) = F_1(g_j \cdot y, g_i \cdot y)$ and $\left|g_i g_j^{-1}\right| = 2$.

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Corollary

Define $\alpha = |\{g \in S | \, |g| = 2\}|$ and
$\beta = \left|\left\{g_i g_j^{-1} | g_i, g_j \in S, \left|g_i g_j^{-1}\right| = 2, \text{ and } i < j\right\}\right|$. Then

$$|F_1(\Delta)| = \left(\binom{n+1}{2} - \frac{1}{2}(\alpha + \beta)\right)|G|$$

$H_n = 0$ for n Generators

Theorem

Let S be a irredundant generating set of size n for G. Then $\widetilde{H_n}\left(N_{\{0,1\}}\left(Cayley(G,S)\right)\right) = 0$ *except when*

1. $n = 1$ *in which case G is cyclic and thus* $\widetilde{H_1} = \mathbb{K}$*, or*
2. $n = 2$ *and G is the Klein 4-group in which* $\widetilde{H_2} = \mathbb{K}$*.*

Group Action

G acting on Cayley graph induces:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathbb{K}^{|F_n(\Delta)|} & \xrightarrow{\partial_n} & \dots & \xrightarrow{\partial_2} & \mathbb{K}^{|F_1(\Delta)|} & \xrightarrow{\partial_1} & \mathbb{K}^{|F_0(\Delta)|} & \xrightarrow{\partial_0} & \mathbb{K} & \longrightarrow & 0 \\
 & & \downarrow \sigma_n & & \downarrow & & \downarrow \sigma_1 & & \downarrow \sigma_0 & & \downarrow & & \\
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\end{array}
$$

- σ_i is a matrix of $0, \pm 1$
- Let α be a word in $F_i(\Delta)$, e_α be the corresponding basis vector in $\mathbb{K}^{|F_i(\Delta)|}$ and ρ the permutation on the labeling. Then $\sigma_i(e_\alpha) = \text{sgn}(\alpha) e_{\rho(\alpha)}$

Note: $\text{sgn}(\alpha)$ is defined to be the parity of the permutation which restores the elements of α to ascending order.

Representation Theory Review

Definition

A homomorphism from a group G to general linear group $GL_n(\mathbb{K})$ over $\mathbb{K}$ of some degree n is a matrix representation of G of degree n.

Definition

An irreducible representation is a representation with no nontrivial invariant subspaces.

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Theorem

The regular representation contains every irreducible representation i times where i is the degree of the irreducible representation.

The Action with $G = \langle g_1, g_2 \rangle$

The action of G on the Cayley graph induces an action on the chain complex and therefore on $F_i(\Delta)$ for $0 \leq i \leq 2$.

The action on $F_0(\Delta)$ and $F_2(\Delta)$ gives the regular representation.

Focus on action of G on $F_1(\Delta)$.
Orbits:
$E_{g_1} \doteq \{\mathrm{F}_1\left(v, g_1 \cdot v\right) | v \in V(\mathrm{Cayley}\left(G, \{g_1, g_2\}\right))\}$
$E_{g_2} \doteq \{\mathrm{F}_1\left(v, g_2 \cdot v\right) | v \in V(\mathrm{Cayley}\left(G, \{g_1, g_2\}\right))\}$
$E_{g_1 g_2^{-1}} \doteq \{\mathrm{F}_1\left(g_1 \cdot v, g_2 \cdot v\right) | v \in V(\mathrm{Cayley}\left(G, \{g_1, g_2\}\right))\}$

Description of Action on F_1

Theorem

Let $\{g_1, g_2\}$ be an irredundant generating set for a group G. Define the set $\mathcal{G} = \{g_1, g_2, g_1 g_2^{-1}\}$. The representation given by the group action on the set of edges, $F_1(\Delta)$, consists of

- *one copy of the regular representation for each element in $\mathcal{G}$ which is not order two*
- *a half of the regular representation for each element in $\mathcal{G}$ which has order two.*

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Moreover for $a \in \mathcal{G}$ of order two, the constituents of a half of the regular representation are given using the formula:

$$\left(\chi, \rho\uparrow^G\right) = (\chi\downarrow_{\langle a\rangle}, \rho) = \frac{1}{|\langle a\rangle|}\sum_{x\in\langle a\rangle}\chi(x)\overline{\rho(x)} = \frac{1}{2}\left(\chi(1) - \chi(a)\right)$$

Working towards the Description of Action on F_1

Lemma

$\{F_1(v, g_i \cdot v), F_1(g_i \cdot v, v)\}$ *defines a block system for* E_{g_i} *if* $|g_i| = 2$.

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Lemma

$\{F_1(v, g_i \cdot v), F_1(g_i \cdot v, v)\}$ *defines a block system for* E_{g_i} *if* $|g_i| = 2$.

Lemma

$Stab_G(\{F_1(1, g_i \cdot 1), F_1(g_i \cdot 1, 1)\}) = \langle g_i \rangle$ *if* $|g_i| = 2$.

Working towards the Description of Action on F_1

Lemma

$\{F_1(v, g_i \cdot v), F_1(g_i \cdot v, v)\}$ defines a block system for E_{g_i} if $|g_i| = 2$.

Lemma

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Lemma

$\{F_1(g_i \cdot v, g_j \cdot v), F_1(g_j \cdot v, g_i \cdot v)\}$ defines a block system for $E_{g_i g_j^{-1}}$ if $\left|g_i g_j^{-1}\right| = 2$.

Lemma

$Stab_G(\{F_1(g_i \cdot 1, g_j \cdot 1), F_1(g_j \cdot 1, g_i \cdot 1)\}) = \left\langle g_i g_j^{-1} \right\rangle$ if $\left|g_i g_j^{-1}\right| = 2$.

Description of Action on F_1

Theorem

Let $\{g_1, g_2\}$ be an irredundant generating set for a group G. Define the set $\mathcal{G} = \{g_1, g_2, g_1 g_2^{-1}\}$. The representation given by the group action on the set of edges, $F_1(\Delta)$, consists of

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Example: Symmetric Group on Four Points

$$\langle g_1 \doteq (1,2), g_2 \doteq (1,2,3,4) \rangle = S_4$$

$$\left(\chi, \rho \uparrow^G\right) = (\chi \downarrow_{\langle a \rangle}, \rho) = \frac{1}{|\langle a \rangle|} \sum_{x \in \langle a \rangle} \chi(x)\overline{\rho(x)} = \frac{1}{2}(\chi(1) - \chi(a))$$

	1A	2A	3A	2B	4A
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	-1	1
χ_5	3	-1	0	1	-1

Table: Character Table for S_4.

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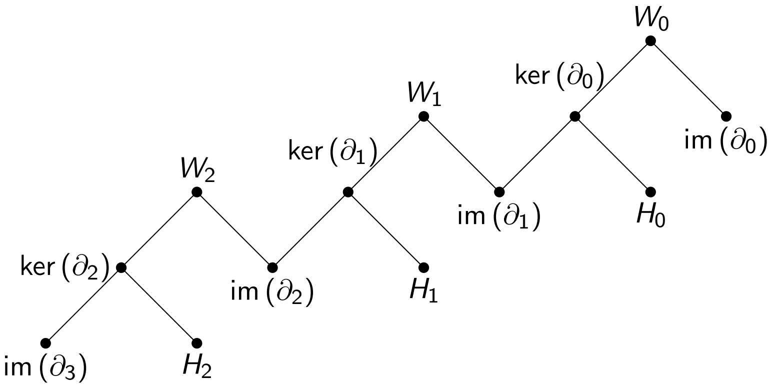
Table: Character Table for S_4.

Copies of	irreducible rep.
2	χ_1
3	χ_2
5	χ_3
8	χ_4
7	χ_5

Note that g_1 in the conjugacy class labeled 2B in the above table.

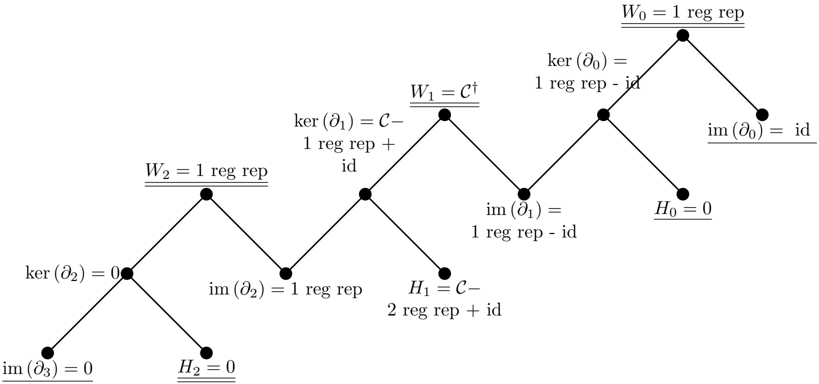
Organizational Structure



Meets are relations:
$\operatorname{im}(\partial_{i+1}) + H_i = \ker(\partial_i)$ and $\ker(\partial_i) + \operatorname{im}(\partial_i) = W_i$

Apply Organizational Structure

$$0 \longrightarrow \mathbb{K}^{|F_2(\Delta)|} \xrightarrow{\partial_2} \mathbb{K}^{|F_1(\Delta)|} \xrightarrow{\partial_1} \mathbb{K}^{|F_0(\Delta)|} \xrightarrow{\partial_0} \mathbb{K} \longrightarrow 0$$



†See Theorem 4.4 for value of $\mathcal{C}$.

Organizational Structure Implies...

Corollary

Let $\{g_1, g_2\}$ be an irredundant generating set for a group G. Let $\mathcal{C}$ be the collection of irreducible representations given by the action on the set of edges $F_1(\Delta)$. Then the irreducible representations given by the action of G on

- *$ker(\partial_1)$ is $\mathcal{C}$ minus one regular representation of G plus the trivial representation of G*
- *$\widetilde{H_1}$ is $\mathcal{C}$ minus two copies of the regular representation of G plus the trivial representation of G.*

Back to Symmetric Group Example

By Corollary 4.10 we can complete the following table of counts of irreducible representations.

- $\ker(\partial_1)$ is $\mathcal{C}$ minus one regular representation of G plus the trivial representation of G
- $\widetilde{H_1}$ is $\mathcal{C}$ minus two copies of the regular representation of G plus the trivial representation of G.

irred rep	Degree	$\mathrm{im}(\partial_2)$	$\ker(\partial_1)$	$\ker(\partial_1)/\mathrm{im}(\partial_2)$	$F_1(\Delta) = \mathcal{C}$
χ_1	1	1	2	1	2
χ_2	1	1	2	1	3
χ_3	2	2	3	1	5
χ_4	3	3	5	2	8
χ_5	3	3	4	1	7

More than Two of Generators

Theorem

Let $S = \{g_1, g_2, ..., g_n\}$ be an irredundant generating set for a group G. Define the set $\mathcal{G} = S \cup \left\{ g_i g_j^{-1} | g_i, g_j \in S \text{ and } i < j \right\}$. The representation given by the group action on $F_1(\Delta)$ consists of

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References

- The GAP Group, *Gap – groups, algorithms, and programming, version 4.6.2*, 2013.
- K. Lux and H. Pahlings, *Representations of groups*, first ed., Cambridge University Press, 2010.
- E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, first ed., Springer, 2005.
- *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org/A135278.

Thank you.

Questions?