Growth in Baumslag-Solitar Groups
Asymptotics

Groups St Andrews
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This work is motivated by an open question posed by deLaHarpe in his book *Topics in Geometric Group Theory* namely, compute the growth series of the Baumslag-Solitar group BS(2,3) using the usual presentation.

Several prominent group theorists have expended considerable effort on this problem without making substantial progress. Evidently it is a difficult problem.

When confronting a hard problem, one approach is reductionism: **divide and conquer**. We use such an approach in concert with **geometry** and **formal language theory**.
A growth series (of a finitely generated group) is a formal generating function:

Summing over $n \geq 0$, let $f(z) = \sum \sigma_n z^n$

where $\sigma_n$ denotes the number of group elements whose (reduced) word length is $n$. Sometimes $f(z)$ is a quotient of polynomials........

“Nice” groups (hyperbolic, automatic, etc) have such rational growth functions. So do solvable Baumslag-Solitar groups (1994 Collins, Edjvet, Gill; 1994 Brazil) and automatic Baumslag-Solitar groups (1992 Edjvet, Johnson).

The **exponent of growth** is the limsup of $n^{th}$ root of $\sigma_n$. This is the reciprocal of the smallest singularity for $f(z)$. For group growth, the above limit exists via Fekete’s Lemma.
Standard presentation for BS(p,q)

\[ \langle b, t \mid tb^p t^{-1} = b^q \rangle \]

(we only consider 1<p<q)

Geometry: basic relator for BS(p,q) is called a “horobrick”. Here p=2, q=3.
Horobricks are glued to form “sheets”
( BS(2,4) in this slide )
Main upper half sheet for BS(2,4)
and typical “mesa” appearing McCann geodesic.
(each sheet is quasi-isometric to the hyperbolic plane)
Main upper half sheet for BS(2,3) and typical McCann geodesic (note lack of horizontal periodicity)
Languages have no formal inverses, so we use $B$ and $T$ as the letters mapping to the inverses of $b$ and $t$, respectively.
Other sheets branch off the main coset: $q$ upward and $p$ downward. Pictured is BS(2,4).
Glue upper and lower half sheets to realize the (partial) Cayley complex: this is BS(2,6) (homeomorphic to tree cross line)
The convolution idea: (Cauchy product) discover the relative growth on each horocycle type; count how many such horocycles are at distance $r$ from the origin for each $r$; sum.

(Here the group is Higman 3, and $r=4$ for the blue coset)
Growth along horocyclic cosets was discussed in an earlier paper (2010 Freden, Knudson, Schofield) where the following were shown:

- When $p$ divides $q$, horocyclic growth is rational and computed
- When $p$ fails to divide $q$, horocyclic growth is nasty
  - No geodesic combing by any context-free language
  - Mathematica suggests horocyclic growth function is transcendental
  - Computation appears NP-hard
Overall growth problem seems too hard!

How about finding the asymptotics? The convolution idea is used to prove that the overall exponent of growth for BS(p,q) is the greater of:

- Growth exponent along horocyclic subgroup (fiber)
- **Growth exponent of branching (base)**
Bass-Serre tree results from projection of the Cayley graph [BS(2,6) again] showing several edge weights. If we know all the weights, we can compute the growth of the tree.
For the solvable groups BS(1,n) and automatic groups BS(n,n) the edge weight schemes are sufficiently simple to permit direct calculation of the branching growth (and indeed, the branching growth function is a proper factor of the overall group growth).

Let's examine the next easiest case, namely BS(2,4). The edge weightings are determined by the geodesic paths $t$, $bt$, $Bt$, $bbt$, $BT$, $T$.

Suppose we're given a horocycle. Define a *relative origin* for this coset as a closest point project on the coset to the absolute origin of the Cayley graph.
Illustration of relative origins and their geodesic paths in the Cayley graph of BS(2,4)

- Horocycles can have *many* relative origins, arranged in diverse ways with many geodesic paths back to the absolute origin
- Look at the figure sideways so that each horocycle projects to a point
- Our edge paths $BT$, $bbt$, etc. in the Cayley graph of the group are combined as *equivalence classes* in the Bass-Serre tree
There are four different branch type of nodes, depending on the arrangement of relative origins found in the underlying coset. This figure shows:

- The branching at the node in the Bass-Serre tree depends upon the arrangement of relative origins in the underlying coset
- There are several coset types (three of four shown here)
The four branch types for BS(2,4)

- Origin type (occurs only once)
- \( \alpha \)-type is characterized by
  - adjacent relative origins are at least distance 4 apart
  - any geodesic path from the absolute origin to any of the relative origins ends in \( t \)
- \( \beta \)-type is characterized by
  - some adjacent relative origins are at distance 2 apart
  - geodesic path from the absolute origin to any of the relative origins end in \( t \) (actually superfluous for \( \beta \)'s)
- \( \gamma \)-type is like \( \alpha \)-type but paths from origin end in \( T \)
BS(2,4) origin type and the similar $\alpha$-type in the projected Bass-Serre tree view
β-type and γ-type
(observe the edge weights)
The $\alpha$, $\beta$, $\gamma$ branch types for BS(2,4)

- Are actually truncated cone types in the Bass-Serre tree
  - (in fact, there are many cone types in this system)
    - but only the $\alpha$, $\beta$, $\gamma$ types if we limit ourselves to the star of a vertex
  - how do these cones and edges relate?
  - are there rules for assembly?
  - (for BS(1,n) and BS(n,n) there are only finitely many rules: easy recursions)
If we know each type, we know the edge weight assignments, and can do the growth computation:

- Each $\alpha$-type spawns
  - 1 node at distance 1 (edge path is $t$)
  - 3 nodes at distance 2 (paths are $bt$, $Bt$, and $BT$)
  - 1 node at distance 3 (path is $bbt$)

- Each $\beta$-type spawns
  - 2 nodes at distance 1 (edge paths are $t$ and $u$)
  - 3 nodes at distance 2 (paths are $bt$, $BT$, and $BT$)

- Each $\gamma$-type spawns
  - 1 node at distance 1 (edge path is $T$)
  - 3 nodes at distance 2 (paths are $bt$, $Bt$, and $BT$)
  - 1 node at distance 3 (path is $bbt$)

- Edge paths $T$, $BT$ result in new $\gamma$-type; $bbt$ in new $\beta$-type
These recursions yield the following equations

- \[ T(z) = A(z) + B(z) + \Gamma(z) + 1 \]

- \[ T(z) = (z + 3z^2 + z^3)\{A(z) + \Gamma(z)\} + (2z + 3z^2)B(z) + 1 + 5z^2 \]

- \[ \Gamma(z) = z\Gamma(z) - z + z^2T(z) \]

Where \( T(z) \) is the growth series for the entire tree

- \( A(z) \) is the growth series for all alpha type vertices in the tree
- \( B(z) \) is the growth series for all beta type vertices in the tree
- \( \Gamma(z) \) is the growth series for all gamma type vertices in the tree
These equations imply

- The growth series for any of the alphas, betas, or gammas is rational (resp. algebraic) if and only if the other two are rational (resp. algebraic).

- Unfortunately the system of equations is underdetermined and needs further information (in the form of relative origin arrangements to determine when β-types occur).

- The need for further information can be tabularized.
The “concatenation table” for tree language in BS(2,4)

- There are rules for assembly! Happy faces allow for concatenation. Frowny faces disallow concatenation.
- The question marks require further information in order to make the concatenation decision.

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<th>t</th>
<th>Bt</th>
<th>bt</th>
<th>bbt</th>
<th>u</th>
<th>T</th>
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The Necessary Further Information

Since various types are completely determined by relative origin arrangements (and direction back to the absolute origin), perhaps we can encode the relative origin arrangements?

First idea: list gaps between relative origins on a horocycle from left to right, so the encoding .2.6.2. refers to the beta type below

Better idea (due to Jared Adams): exploit the doublings, halvings, and overlaps and record in a **Q-code**. The code for the above coset is $101Q$ which translates to the *unique ternary integer* 10.

Q-codes yield good compression, with ternary even better: .2.2.2.2.2.2.2.2.2.2.2.2.2.2.2.2.2.2.2.2. stores as $1212Q$ or 50
Example

- The above $\beta$-type coset (vertex in the tree) has Q-code $101Q$ or $10$
- There are four upward and one downward edges to be attached:

(diagram above) $\text{Bt}$ leads to $\beta$-type $11Q$ or $4$

(diagram above) $t$, $bt$, and $u$ all lead to $\alpha$-type $10Q$ or $3$

(diagram above) $\text{BT}$ leads to $\gamma$-type $1020Q$ or $33$
BS(2,4) encodings
We have proved:

- Each vertex in the Bass-Serre tree of BS(2,4) has its unique Q-code (which we interpret as an integer via ternary expansion)
- Each Q-code defines a unique cone type in the tree
- There is a bijection between origin arrangements as Q-codes and the integers 0, 1, 2, 3, …
- Consequently there are infinitely many cone types in the tree
- Each cone contains *every other* cone type (and infinitely often)
- Consequently each cone generating function can be expressed as a rational function of any other cone OGF (along with other stuff)
- The growth series $T(z)$ for the entire tree contains infinitely many singularities and is not D-finite
- The growth exponent is estimated between 2.5014 and 2.56
Experimental Math: calculate 36 terms of $T(z)$, do diagonal Pade approximant, plot poles
Summary of ideas

- The Q-codes completely determine all edge weight assignments
- The edge weight assignments determine the growth series $T(z)$
- Hypothesis: there is no nice recursion for Q-code computations
  - Show that such a hypothesized recursion yields a polynomial solution to
    general integer factoring or the 3n+1 Collatz problem?
- If so, only brute-force is available
  - our algorithm has time & space complexity about $\Theta(1.68^r)$; not in PSPACE
- Comparison with other methods?
  - Diekert & Laun have a polynomial algorithm converting words to geodesics
    Input Knuth-Bendix-Thurston (KBT) normal forms for Bass-Serre tree, output
    geodesic stems, keep counts?
  - No good! KBT normal forms are exponentially long and there are
    exponentially many words to count; time complexity about $\Theta(10.73^r)$ although
    space complexity appears linear.
Formal language aspects

- Jared Adams has written software that
  - on input of a valid geodesic stem path outputs the stack code for each coset the path goes through (polynomial time/space)
  - on input of any non-valid stem path outputs an error message
  - accepts only ONE stem path for any given terminus coset
  - (update: there is now a spreadsheet version!)

- This defines a *combing* for the Bass-Serre tree of BS(2,4)

- What are the properties of this formal language?
  - not context-free language (fails Ogden’s lemma; also the OGF is not D-infinite, hence not algebraic)
  - probably not indexed because the stack codes require arithmetic not possible on a nested stack automaton
Future goals

- Determine the language class of the BS(2,4) tree combing
- Prove an exponential lower bound for its growth computation
- Show the coefficients of $T(z)$ form a supermultiplicative sequence
- Show the exponent of growth $\omega$ is transcendental
- (Unpleasant thought: just because $\omega$ is hard to compute via $T(z)$ does not imply that $\omega$ is hard to compute via another method)
- Extend the methods to BS(2,2k) any $k>2$ (appears feasible)
- Extend to BS(p,q) for any p divides q (appears formidable)
- Think about same when p fails to divide q (looks impossible)
When $p$ fails to divide $q$, weird and complicated things happen. Relative origins are in red; look at the left side of this lower half sheet in BS(2,3).

The relative origin appears out of the blue with NO geodesic path to any previous relative origin! This precludes any encoding scheme based on relative origin arrangements.
Thanks to the organizers and audience!

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