A non-embedding result for Thompson’s Group V

Nathan Corwin

University of Nebraska – Lincoln

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s-ncorwin1@math.unl.edu
Overview

Theorem (C. 2013)

\[ \mathbb{Z} \wr \mathbb{Z}^2 \] does not embed into Thompson’s Group V.
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- Give some motivation.

Introduction
- coCF groups
- Wreath Products
- Thompson’s Group V
- Dynamics of V
- Proof of Main Result
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- Briefly discuss the proof of the theorem.
In the mid-1980's Muller and Schupp showed that the class of all of groups that have a context free word problem (denoted $\text{CF}$) is equivalent to the class of groups that are virtually free.

A natural generalization of $\text{CF}$ is the class $\text{coCF}$: all groups for which the coword problem is context free. This class was first introduced by Holt, Rees, Reiner, and Thomas in 2006. They showed that $\text{coCF}$ has many closure properties.

Closed under:
- direct products;
- standard restricted wreath products where the top group is $\text{CF}$;
- passing to finitely generated subgroups;
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Two conjectures from that paper:

1. If $C \wr T$ is in $\text{co}^1\mathcal{F}$, then $T$ must be in $\mathcal{F}$; My theorem supports this conjecture.
2. $\text{co}^1\mathcal{F}$ is not closed under free products. The leading candidate to show the second conjecture is the group $\mathbb{Z} \ast \mathbb{Z}/2$. 
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It also put into doubt the belief that $\mathbb{Z} \ast \mathbb{Z}^2$ is not in $\text{co}C\mathcal{F}$ as $V$ contains many copies of $\mathbb{Z}$ and $\mathbb{Z}^2$ and free products of subgroups are common in $V$. 
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Other motivation
Structure of the R. Thompson’s groups

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- In 2008, Bleak showed that $\mathbb{Z} \wr \mathbb{Z}^2$ does not embed into $F$. 
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Structure of the R. Thompson’s groups

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- In 1999, Guba and Sapir showed that \( \mathbb{Z} \wr \mathbb{Z} \) embeds into \( F \), and thus embeds into \( T \) and \( V \) as well.
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- In 2009, Bleak, Kassabov, and Matucci showed \( \mathbb{Z} \wr \mathbb{Z}^2 \) does not embed into \( T \).
Some notation

Suppose that $a, b \in G$ and $G$ acts on a set $X$. 
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- We write \((x)a\) or just \(xa\) instead of \(a(x)\).
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- Support of a function (element): $\text{Supp } (a) = \{ x \in X | xa \neq x \}$. Note, this differs slightly from the standard analysis definition.
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- Support of a function (element): $\text{Supp} \ (a) = \{x \in X | xa \neq x\}$.
  Note, this differs slightly from the standard analysis definition.
- Fact: $\text{Supp} \ (a^b) = \text{Supp} \ (a)b$. 
Wreath Products

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Then, the Wreath Product of $A$ and $T$ is $A \wr T = B \rtimes T$ (where the semi-direct product action of $T$ on $B$ is right multiplication on the index in the direct product).

We say $T$ is the top group, $A$ is the bottom group, and $B$ is called the base group.
First look at R. Thompson’s group $V$

- $V$ is finitely presented.
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- Standard presentation has 4 generators,
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- The generators of the standard presentation are $A, B, C, \pi_0$.
- The relations:
First look at R. Thompson’s group V

- $[AB^{-1}, A^{-1}BA] = 1$;
- $[AB^{-1}, A^{-2}BA^2] = 1$;
- $C = BA^{-1}CB$;
- $A^{-1}CBA^{-1}BA = BA^{-2}CB^2$;
- $CA = (A^{-1}CB)^2$;
- $C^3 = 1$;
- $((A^{-1}CB)^{-1} \pi_0 A^{-1}CB)^2 = 1$;
- $[(A^{-1}CB)^{-1} \pi_0 A^{-1}CB, A^{-2}(A^{-1}CB)^{-1} \pi_0 A^{-1}CBA^2] = 1$;
- $(A^{-1}CB)^{-1} \pi_0 A^{-1}CBA(A^{-1}CB)^{-1} \pi_0 A^{-1}CB)^3 = 1$;
- $[A^{-2}BA^2, (A^{-1}CB)^{-1} \pi_0 A^{-1}CB] = 1$;
- $(A^{-1}CB)^{-1} \pi_0 A^{-1}CBA^{-1}BA = BA^{-1}(A^{-1}CB)^{-1} \pi_0 A^{-1}CBA(A^{-1}CB)^{-1} \pi_0 A^{-1}CB$;
- $A^{-1}(A^{-1}CB)^{-1} \pi_0 A^{-1}CBA = BA^{-2}(A^{-1}CB)^{-1} \pi_0 A^{-1}CBA^2$;
Second look at R. Thompson’s group $V$

Let $\mathcal{T}$ be the infinite, rooted, directed, binary tree.
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Note that the limit space is the Cantor set.
An element of $V$

Let $D$ and $R$ be two finite connected rooted subgraphs of $T$ (with the same root as $T$) both with $n$ leaves for some arbitrary $n$. Let $\sigma \in S_n$. Then $u = (D,R,\sigma)$ is a representative of an element of $V$. (Tree pair representative)
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Example of an element $U$ in $V$
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Example of an element $U$ in $V$

$(0101100\ldots)U$
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$$(0101100 \ldots)U = 101100 \ldots$$
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Not unique representation
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A simplifying assumption for this talk

I will assume that every element of $V$ (I discuss) has no non-trivial orbits when it acts on the Cantor set.
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- This is not a big assumption (for me) as for any $v \in V$ there is an $n \in \mathbb{N}$ such that $v^n$ has this condition.
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- I will assume that every element of $V$ (I discuss) has no non-trivial orbits when it acts on the Cantor set.
- This is not a big assumption (for me) as for any $v \in V$ there is an $n \in \mathbb{N}$ such that $v^n$ has this condition.
- This assumption is not needed to understand the dynamics of $V$, but makes things simpler to explain.
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Revealing Pairs

Consider the common tree $C = D \cap R$. 
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Revealing Pairs and Important points

- A revealing pair is a particularly tree pair of an element of $V$.  

Fact: $\text{Supp}(a) = \text{Supp}(a) \cup I(a)$.  

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Revealing Pairs and Important points

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- The set of the attracting and repelling fixed points are the set of important points for $u$. This set is denoted by $I(u)$.
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Fact: $\text{Supp} (a) = \text{Supp} (a) \cup I(a)$. 
Flow graph

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Components of support
Lemma (Bleak, Salazar-Díaz, 2009)

Suppose $g, h \in V$, each with no non-trivial periodic orbits. For (i) and (ii), suppose further that $g$ and $h$ commute. Then:

i. $I(g) \cap I(h) = I(g) \cap \text{Supp}(h) = I(h) \cap \text{Supp}(g)$;

ii. If $X$ and $Y$ are components of support of $g$ and $h$ respectively, then $X = Y$ or $X \cap Y = \emptyset$;

iii. Suppose $g$ and $h$ have a common component of support $X$, and on $X$ the actions of $g$ and $h$ commute. Then, there are non-trivial powers $m$ and $n$ such that $g^m = h^n$ over $X$. 
Proof (outline) of main result

Recall

Theorem (C. 2013) \( \mathbb{Z} \wr \mathbb{Z}^2 \) does not embed into Thompson’s Group V.
Recall

**Theorem (C. 2013)**

$\mathbb{Z} \wr \mathbb{Z}^2$ does not embed into Thompson’s Group $V$.

Proof:

- **Step 1:** Suppose there is an injection $\phi : \mathbb{Z} \wr \mathbb{Z}^2 \to V$
Recall

**Theorem (C. 2013)**

\[\mathbb{Z} \wr \mathbb{Z}^2 \text{ does not embed into Thompson's Group } V.\]

**Proof:**

- **Step 1:** Suppose there is an injection \( \phi : \mathbb{Z} \wr \mathbb{Z}^2 \to V \)
- **Step 2:** Clean up injection
Let $s'$ and $t'$ be the images of the generators of the $\mathbb{Z}^2$. 
Proof sketch continued

Step 2: Clean up injection

- Let $s'$ and $t'$ be the images of the generators of the $\mathbb{Z}^2$.
- Raise $s'$ and $t'$ to powers to obtain $s$ and $t$ with no non-trivial finite orbits.
Proof sketch continued
Step 2: Clean up injection

- Let $s'$ and $t'$ be the images of the generators of the $\mathbb{Z}^2$.
- Raise $s'$ and $t'$ to powers to obtain $s$ and $t$ with no non-trivial finite orbits.
- Fix an element $\gamma_0'$ in the bottom group with no non-trivial finite orbits.
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- Repeatedly apply two technical lemmas of Bleak and Salazar-Díaz to eventually replace $\gamma'_0$ with $\gamma_0$. 
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- $\gamma_0$ will:
  - be a non-trivial element of the base with no non-trivial finite orbits;
  - have support disjoint from a neighborhood of the important points of $s$ and $t$.
Proof sketch continued

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$\gamma_0$ will:
- be a non-trivial element of the base with no non-trivial finite orbits;
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We have $\langle s, t, \gamma_0 \rangle \cong \mathbb{Z} \wr \mathbb{Z}^2$. 
Proof:

- **Step 1** Suppose there is an injection $\phi : \mathbb{Z} \wr \mathbb{Z}^2 \to V$
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Proof sketch continued

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- Step 1: Suppose there is an injection $\phi : \mathbb{Z} \wr \mathbb{Z}^2 \to V$
- Step 2: Clean up injection
- Step 3: Make sequence of $\gamma_i$'s
Proof sketch continued

Step 3: Make sequence of $\gamma_i$’s

- Consider the subgroup $\langle s, r \rangle$. It has components of support $X_1, \ldots, X_k$. 

On $X_1$, $s$ and $r$ commute, so there are integers $r, q \neq 0$ such that $u = sr^tq$ is trivial on $X_1$. Thus, there is a power $p$ such that $\text{Supp}(u) \cap \text{Supp}(\gamma_0) \cap \text{Supp}(\gamma_0)^u = \emptyset$. Set $w = u^p$. Define $\gamma_1 = [\gamma_0, w]$. This element is nontrivial and of infinite order and will have no important points in $X_1$. One can show that $\langle s, t, \gamma_1 \rangle \sim \mathbb{Z} \rtimes \mathbb{Z}_2$. 

Proof sketch continued

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- Thus, there is a power $p$ such that $\text{Supp}(u) \cap \text{Supp}(\gamma_0) \cap \text{Supp}(\gamma_0 u^p) = \emptyset$. Set $w = u^p$.
- Define $\gamma_1 = [\gamma_0, w]$. This element is nontrivial and of infinite order and will have no important points in $X_1$.
- One can show that $\langle s, t, \gamma_1 \rangle \cong \mathbb{Z} \wr \mathbb{Z}^2$. 
Proof sketch continued
Step 3: Make sequence of $\gamma_i$'s

- Recursively repeat this process: given a $\gamma_{i-1}$, we can make a $\gamma_i$. 

Each time, we have $\gamma_i$ is nontrivial and of infinite order. Further, $\langle s, t, \gamma_i \rangle \sim \mathbb{Z} \rtimes \mathbb{Z}^2$. These elements were made so that $\gamma_i$ has no important points in $X_i$. One can show that $\gamma_i$ has no important points in $X_j$ for $j < i$. In particular, $\gamma_k$ has no important points at all, thus it is trivial.
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Proof:

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- Step 2: Clean up injection
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- Step 4: Show that $\gamma_k$ is a nontrivial element of infinite order that is also trivial
- Step 5: Notice that this is a contradiction
Finish proof sketch

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Thank you for your attention.