A generalisation on the solvability of finite groups with three class sizes for normal subgroups

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(in collaboration with María José Felipe)
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New Topic
Influence of $cs_G(N)$ on the structure of $N$
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**Example**

Let $G = S_3 \rtimes \mathbb{Z}_2$ and let $N = S_3 \times S_3 \trianglelefteq G$. Then

$$cs(N) = \{1, 2, 3, 4, 6, 9\}, \text{ while } cs_G(N) = \{1, 4, 6, 9, 12\}.$$
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If $cs(G) = \{1, m\}$ then $m = p^a$ for some prime $p$ and $G = P \times A$, with $A$ abelian, and $P$ a $p$-group.
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**Theorem (Alemany, Beltrán, Felipe)**

Suppose that $N$ is a normal subgroup of a group $G$ having two $G$-class sizes, then either $N$ is abelian or $N = P \times A$, with $P$ a $p$-group and $A \subseteq Z(G)$.

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**Definition.** A nonabelian group $G$ is said to be an F-group if for every $x, y \in G \setminus Z(G)$, such that $C_G(x) \subseteq C_G(y)$, then $C_G(x) = C_G(y)$.

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- F-groups were classified by J. Rebmann (1971). As a consequence, Rebmann obtains the solvability of groups with three class sizes which are F-groups.
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- A. Camina (1974) shows that any group with three class sizes which is not an F-group is a direct product of an abelian group and a group whose order involves no more than two primes.
Theorem (Dolfi, Jabara, 2009)

A finite group $G$ has three class sizes if and only if, up to an abelian factor, either

1. $G$ is a $p$-group for some prime $p$ or
2. $G = KL$ with $K \leq G$, $(|K|, |L|) = 1$ and one of the following occurs
   a. both $K$ and $L$ are abelian, $Z(G) < L$ and $G$ is a quasi-Frobenius group,
   b. $K$ is abelian, $L$ is a non-abelian $p$-group, for some prime $p$ and $O_p(G)$ is an abelian subgroup of index $p$ in $L$ and $G/O_p(G)$ is a Frobenius group or
   c. $K$ is a $p$-group with two class sizes for some prime $p$, $L$ is abelian, $Z(K) = Z(G) \cap K$ and $G$ is quasi-Frobenius.
Solvability of $N \trianglelefteq G$ with $\text{cs}_G(N) = \{1, m, n\}$

The case in which $m$ does not divide $n$

Definition

A non-central normal subgroup $N$ of a group $G$ is said to be an $F$-normal subgroup if for every $x, y \in N \setminus Z(G)$, such that $C_G(x) \subseteq C_G(y)$, then $C_G(x) = C_G(y)$.

Lemma

If $N$ is an $F$-normal subgroup of a group $G$, then $N/(N \cap Z(G))$ has a non-trivial normal abelian partition.

We use results of Baer and Suzuki on groups having a non-trivial normal partition to classify $F$-normal subgroups.
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The case in which $m$ does not divide $n$

**Theorem (Akhlaghi, Beltrán, Felipe, Khatami)**

Let $G$ be a group and $N$ be an F-normal subgroup of $G$. Then $N$ satisfies one of the following conditions:

1. $N/Z(N)$ is a Frobenius group, with Frobenius kernel $L/Z(N)$ and complement $K/Z(N)$, with $K$ and $L$ abelian.

2. $N/Z(N)$ is a Frobenius group, with kernel $L/Z(N)$ and complement $K/Z(N)$, where $K$ is abelian, and $L/Z(N)$ is of prime-power order, and $L$ is an F-normal subgroup.

3. $N/Z(N) \cong S_4$ and $V$ is non-abelian, for $V/Z(N)$, the Klein four-group of $N/Z(N)$. In particular, $N$ is an F-group.

4. $N$ has abelian Fitting subgroup of index $p$, $p$ divides $|F(N)/Z(N)|$, and $N$ is an F-group.

5. $N = P \times Z(N)_{p'}$, where $P \in Syl_p(N)$.

6. $N/Z(N) \cong \text{PSL}(2, p^h)$ or $\text{PGL}(2, p^h)$, where $p^h \geq 4$. 
The case in which $m$ does not divide $n$

Theorem (Akhlaghi, Beltrán, Felipe, Khatami)

Let $N$ be an $F$-normal subgroup of $G$ such that $|\text{cs}_G(N)| = 3$. Then $N$ is solvable. In particular, when $\text{cs}_G(N) = \{1, m, n\}$ and $m$ does not divide $n$, then $N$ is solvable.
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The proof consists in showing that case (6) in the above classification cannot happen.

Theorem (Akhlaghi, Beltrán, Felipe, Khatami)

Let $N$ be a normal subgroup of a finite group $G$ such that $\text{cs}_G(N) = \{1, m, n\}$, where $m < n$ and $m$ does not divide $n$. Then one of the following conditions is satisfied:

(1) $N = P \times A$, where $P \in \text{Syl}_p(N)$, $p$ prime and $A \subseteq \mathbb{Z}(G)$.

(2) $N/\mathbb{Z}(N)$ is a Frobenius group, with Frobenius kernel $L/\mathbb{Z}(N)$ and Frobenius complement $K/\mathbb{Z}(N)$, and

(a) either $K$ and $L$ are abelian, and

$$\text{cs}(N) = \{1, |L/\mathbb{Z}(N)|, |K/\mathbb{Z}(N)|\}.$$ 

(b) or $K$ is abelian, and $L/\mathbb{Z}(N)$ is of prime-power order, and

$$\text{cs}(N) = \{1, |L/\mathbb{Z}(N)|, |K/\mathbb{Z}(N)||x^L| : x \in L \setminus \mathbb{Z}(N)\}.$$
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     Thus, $C_N(z) \triangleleft C_G(z)$ and this normal subgroup has at most
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Thus, $C_N(z) \trianglelefteq C_G(z)$ and this normal subgroup has at most two $p$-regular $C_G(z)$-class sizes.

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**Theorem (Akhlaghi, Beltrán, Felipe, J. Group Theory, 2013)**

Let $N \triangleleft G$ having exactly two $G$-class sizes of $p$-regular elements. Then $N$ is solvable. Moreover, either $N$ has abelian $p$-complements or all $p$-regular elements of $N/(N \cap Z(G))$ have prime power order.
b) Properties of $F(N)$ and $Z(N)$. 
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**Theorem**

Suppose that $N$ is a solvable normal subgroup of a group $G$ and suppose that an integer $m$ divides $s$ for every $s \in \text{cs}_G(N)$, $s \neq 1$. If $g \in N$ and $|g^G| = m$, then $g \in F(N)$.
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**Theorem**

Suppose that $N$ is a nonsolvable normal subgroup of a group $G$ and suppose that an integer $m$ divides $|x^G|$ for every $x \in N \setminus Z(N)$. Then $m$ divides $|Z(N)|$. 
Lemma

A finite nonabelian simple group does not have a nontrivial conjugacy class whose size divides the order of its Schur multiplier.
The case in which $m$ divides $n$

Lemma
A finite nonabelian simple group does not have a nontrivial conjugacy class whose size divides the order of its Schur multiplier.

Theorem
If $N$ is a nonabelian normal subgroup of a finite group $G$ and $|\text{cs}_G(N)| = 3$, then $Z(N)$ is properly contained in $F(N)$.
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If $N$ is a nonabelian normal subgroup of a finite group $G$ and $|c_{SG}(N)| = 3$, then $Z(N)$ is properly contained in $F(N)$.

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Suppose $N \unlhd G$. What happens if every element in $G \setminus N$ has the same class size?

Theorem (Isaacs, 1970) Let $N$ be a normal subgroup of a group $G$ such that all of the conjugacy classes of $G$ which lie outside $N$ have equal sizes. Then $G/N$ is cyclic or else every nonidentity element of $G/N$ has prime order.
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**Definition**

A normal section $N/K$ of a group $G$ satisfies condition (*) over $G$ when $N$ is a nonabelian normal subgroup of $G$ such that all the $G$-conjugacy classes in $N$ lying outside of $K$ have equal size.

**Theorem (Akhlaghi, Beltrán, Felipe)**

Let $N/K$ be a normal section satisfying (*) over $G$.

i) If $Z(N) \not\subseteq K$, then $N/K$ is a $p$-group for some prime $p$ and $N/K$ is either abelian or has exponent $p$.

ii) If $Z(N) \subseteq K$, then either $N/K$ is cyclic or is a $CP$-group. In the first case, $N$ has abelian Hall $P$-complement, where $P = (N/K)$.

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A generalisation on the solvability of finite groups with three classes...
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d) CP-groups and final arguments.

\[ N/F(N) \] is a non-solvable CP-group.

Theorem (Heineken, 2006)

If \( G \) is a finite non-solvable CP-group, then there exist normal subgroups \( B, C \) of \( G \) such that \( 1 \subseteq B \subseteq C \subseteq G \) and \( B \) is a 2-group, \( C/B \) is non-abelian and simple, and \( G/C \) is a \( p \)-group for some prime \( p \) and cyclic or generalised quaternion. In particular, if \( G \) is a finite non-abelian simple CP-group, then \( G \) is isomorphic to:

- \( L_2(q) \), for \( q = 5, 7, 8, 9, 17 \)
- \( L_3(4) \)
- \( Sz(8) \) or \( Sz(32) \).

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- By induction on $|N|$ we have that $N$ is perfect. If $N' < N$, then $|cs_G(N')| \leq 3$, so $N'$ is solvable and $N$ is solvable as well.
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- Therefore, there exists $B \trianglelefteq N$, such that $N/B$ is simple (CP-group) and $B/F(N)$ is a 2-group.
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- By induction on \( |N| \) we have that \( N \) is perfect. If \( N' < N \), then \( |cs_G(N')| \leq 3 \), so \( N' \) is solvable and \( N \) is solvable as well.
- Therefore, there exists \( B \trianglelefteq N \), such that \( N/B \) is simple (CP-group) and \( B/F(N) \) is a 2-group.
- We make a case-by-case analysis for each of the simple groups, and we finally get a contradiction.
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When dealing with $G$-class sizes and normal subgroups, such structure does not hold.
Thank you for your attention