On Clifford-Fischer Theory

Ayoub Basheer*
School of Mathematics, Statistics & Computer Science, University of KwaZulu-Natal, Pietermaritzburg
Department of Mathematics, Faculty of Mathematical Sciences, University of Khartoum, P. O. Box 321,
Khartoum, Sudan

Jamshid Moori
School of Mathematical Sciences, North-West University, Mafikeng

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Bernd Fischer presented a powerful and interesting technique, known as *Clifford-Fischer theory*, for calculating the character tables of group extensions. This technique derives its fundamentals from the Clifford theory. In this talk we describe the methods of the coset analysis and Clifford-Fischer theory applied to group extensions (split and non-split). We also mention some of the contributions to this domain and in particular of the second author and his research groups including students.
The Character Table of a Group Extension

- Let $\overline{G} = N \cdot G$, where $N \triangleleft \overline{G}$ and $\overline{G}/N \cong G$, be a finite group extension.
- There are several well-developed methods for calculating the character tables of group extensions. For example, the Schreier-Sims algorithm, the Todd-Coxeter coset enumeration method, the Burnside-Dixon algorithm and various other techniques.
- Bernd Fischer [11, 12, 13] presented a powerful and interesting technique, known nowadays as the **Clifford-Fischer Theory**, for calculating the character tables of group extensions. To construct the character table of $\overline{G}$ using this method, we need to have:
  1. the conjugacy classes of $\overline{G}$ obtained through the coset analysis method,
  2. the character tables (ordinary or projective) of the inertia factor groups,
  3. the fusions of classes of the inertia factors into classes of $G$,
  4. the Fischer matrices of $\overline{G}$. 
For each $g \in G$ let $\bar{g} \in \bar{G}$ map to $g$ under the natural epimorphism $\pi : \bar{G} \longrightarrow G$ and let $g_1 = N\bar{g}_1, g_2 = N\bar{g}_2, \cdots, g_r = N\bar{g}_r$, be representatives for the conjugacy classes of $G \cong \bar{G}/N$. Therefore $\bar{g}_i \in \bar{G}, \forall i$, and by convention we take $\bar{g}_1 = 1_{\bar{G}}$.

The method of the coset analysis constructs for each conjugacy class $[g_i]_G, 1 \leq i \leq r$, a number of conjugacy classes of $\bar{G}$. For each $1 \leq i \leq r$, we let $g_{i1}, g_{i2}, \cdots, g_{ic(g_i)}$ be the corresponding representatives of these classes. That is each conjugacy class of $\bar{G}$ corresponds uniquely to a conjugacy class of $G$.

Also we use the notation $U = \pi(\bar{U})$ for any subset $\bar{U} \subseteq \bar{G}$. Thus we have

$$\pi^{-1}([g_i]_G) = \bigcup_{j=1}^{c(g_i)} [g_{ij}]_{\bar{G}}$$

for any $1 \leq i \leq r$. We assume that $\pi(g_{ij}) = g_i$ and by convention we may take $g_{11} = 1_{\bar{G}}$. 

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The coset analysis method can be described briefly in the following steps:

- For fixed $i \in \{1, 2, \cdots, r\}$, act $N$ (by conjugation) on the coset $N\bar{g}_i$ and let the resulting orbits be $Q_{i1}, Q_{i2}, \cdots, Q_{ik_i}$. If $N$ is abelian (regardless to whether the extension is split or not), then $|Q_{i1}| = |Q_{i2}| = \cdots = |Q_{ik_i}| = \frac{|N|}{k_i}$.

- Act $\bar{G}$ on $Q_{i1}, Q_{i2}, \cdots, Q_{ik_i}$ and suppose $f_{ij}$ orbits fuse together to form a new orbit $\Delta_{ij}$. Let the total number of the new resulting orbits in this action be $c(g_i)$ (that is $1 \leq j \leq c(g_i)$). Then $\bar{G}$ has a conjugacy class $[g_{ij}]_{\bar{G}}$ that contains $\Delta_{ij}$ and $|[g_{ij}]_{\bar{G}}| = |[g_i]_{\bar{G}}| \times |\Delta_{ij}|$.

- Repeat the above two steps, for all $i \in \{1, 2, \cdots, r\}$.
Example of Using the Coset Analysis Technique

In [10] we used the coset analysis to compute the conjugacy classes of \( \overline{G} = 2^{1+6}_1 : (3^{1+2}:8):2 \). This is a maximal subgroup, of index 3, in \( 2^{1+6}_1 : 3^{1+2}:2S_4 \), which in turn is the second largest maximal subgroup of the automorphism group of the unitary group \( U_5(2) \).

Using the coset analysis we found that corresponding to the 14 classes of \( G = (3^{1+2}:8):2 \), we obtain 41 conjugacy classes for \( \overline{G} \). For example the group \( G \) has two classes of involutions represented by \( 2_1 \) and \( 2_2 \) with respective centralizer sizes 48 and 12. Corresponding to the class containing \( 2_2 \) we get five conjugacy classes in \( \overline{G} \) with information listed in the following table.

| \([g_i]G\) | \(k_i\) | \(m_{ij}\) | \([g_{ij}]\overline{G}\) | \(o(g_{ij})\) | \(|[g_{ij}]\overline{G}|\) | \(|C_{\overline{G}}(g_{ij})|\) |
|------|------|------|------|------|------|------|
| \(g_3 = 2_2\) | \(k_3 = 9\) | \(m_{31} = 8\) | \(g_{31}\) | 8 | 288 | 192 |
|       |       | \(m_{32} = 8\) | \(g_{32}\) | 8 | 288 | 192 |
|       |       | \(m_{33} = 24\) | \(g_{33}\) | 2 | 576 | 96 |
|       |       | \(m_{34} = 48\) | \(g_{34}\) | 8 | 1728 | 32 |
|       |       | \(m_{35} = 48\) | \(g_{35}\) | 4 | 1728 | 32 |
If $\overline{G} = N \cdot G$ is a group extension, then $\overline{G}$ has action on the classes of $N$ and also on $\text{Irr}(N)$. Brauer Theorem (see [3] for example) asserts that the number of orbits of these two actions are the same.

Let $\theta_1, \theta_2, \cdots, \theta_t$ be representatives of $\overline{G}$–orbits on $\text{Irr}(N)$ and let $\overline{H}_k$ and $H_k$ denote the corresponding inertia and inertia factor groups of $\theta_k$.

In order to apply the Clifford-Fischer Theory, one have to determine the structures of all the inertia or inertia factor groups.

The Clifford Theory (see [3]) deals with the character tables (ordinary or projective) of the inertia groups.
Inertia Factor Groups

- In practise we do not attempt to compute the character table of $H_k$, simply because the character tables of these inertia groups are usually much larger and more complicated to compute than the character table of $\overline{G}$ itself.

- Bernd Fischer suggested to use the character tables of the inertia factor groups $H_k$ together with some matrices, called by him **Clifford matrices** (throughout this talk we refer to them as **Fischer matrices**), to construct the character table of $\overline{G}$.

- Thus we firstly need to determine the structures and the appropriate projective character table of all the inertia factors $H_k$ together with the Fischer matrices.

- One of the biggest challenges in Clifford-Fischer theory is the determination of the type of the character table of $H_k$ (projective or ordinary), which is to be used in the construction of the character table of $\overline{G}$. 
In practice making the right choice of the appropriate projective character table of $H_k$, with factor set $\alpha_k$, might be difficult unless the Schur multipliers of all the $H_k$ are trivial.

Otherwise there will be many combinations (for each $H_k$, there are many projective character tables associated with different factor sets of the Schur multiplier of $H_k$) and one has to test all the possible choices and eliminate the choices that lead to contradictions.

Some partial results on the extendability of characters are given in [3].

Having determined the structures and the appropriate projective character table of $H_k$, with factor set $\alpha_k$ (that is to be used to construct the character table of $\overline{G}$), the next step will be to determine the fusions of the $\alpha_k$—regular classes of $H_k$ into classes of $G$. 
We proceed to define the Fischer matrices, which are so important to calculate the character table of any group extension $\overline{G} = N \cdot G$, $N \triangleleft \overline{G}$.

For each $[g_i]_G$, there corresponds a Fischer matrix $F_i$.

$$[g_{ij}]_G \cap \overline{H_k} = \bigcup_{n=1}^{c(g_{ijk})} [g_{ijkn}]_H \setminus H_k,$$

where $g_{ijkn} \in \overline{H_k}$ and by $c(g_{ijk})$ we mean the number of $\overline{H_k}$–conjugacy classes that form a partition for $[g_{ij}]_G$. Since $g_{11} = 1_G$, we have $g_{11k1} = 1_{\overline{G}}$ and thus $c(g_{11k1}) = 1$ for all $1 \leq k \leq t$.

$$[g_i]_G \cap H_k = \bigcup_{m=1}^{c(g_{ik})} [g_{ikm}]_H \setminus H_k,$$

where $g_{ikm} \in H_k$ and by $c(g_{ik})$ we mean the number of $H_k$–conjugacy classes that form a partition for $[g_i]_G$. Since $g_1 = 1_G$, we have $g_{1k1} = 1_G$ and thus $c(g_{1k1}) = 1$ for all $1 \leq k \leq t$. Also $\pi(g_{ijkn}) = g_{ikm}$ for some $m = f(j, n)$. 
Labeling Columns & Rows of Fischer Matrices

- The top of the columns of $\mathcal{F}_i$ are labeled by the representatives of $[g_{ij}]_G$, $1 \leq j \leq c(g_i)$ obtained by the coset analysis and below each $g_{ij}$ we put $|C_G(g_{ij})|$.

- The bottom of the columns of $\mathcal{F}_i$ are labeled by some weights $m_{ij}$ defined by

$$m_{ij} = [N_G(Ng_i) : C_G(g_{ij})] = |N| \frac{|C_G(g_i)|}{|C_G(g_{ij})|}.$$ 

- To label the rows of $\mathcal{F}_i$ we define the set $J_i$ to be

$$J_i = \{(k, m) | 1 \leq k \leq t, 1 \leq m \leq c(g_{ik}), \ g_{ikm} \ is \ \alpha_k^{-1} \ - \ regular \ class\}.$$ 

- Then each row of $\mathcal{F}_i$ is indexed by a pair $(k, m) \in J_i$. 

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The Fischer Matrix $\mathcal{F}_i$ Corresponds to $[g_i]_G$

For fixed $1 \leq k \leq t$, we let $\mathcal{F}_{ik}$ be a sub-matrix of $\mathcal{F}_i$ with rows correspond to the pairs $(k, 1), (k, 2), \ldots, (k, r_k)$.

Let

$$a_{ij}^{(k,m)} := \sum_{n=1}^{c(g_{ijk})} \frac{|C_G(g_{ij})|}{|C_{H_k}(g_{ijkn})|} \tilde{\psi}_k(g_{ijkn})$$

(for which $\pi(g_{ijkn}) = g_{ikm}$).

For each $i$, corresponding to the conjugacy class $[g_i]_G$, we define the Fischer matrix $\mathcal{F}_i = \left(a_{ij}^{(k,m)}\right)$, where $1 \leq k \leq t$, $1 \leq m \leq c(g_{ik})$, $1 \leq j \leq c(g_i)$. 
The Fischer matrix $F_i$ corresponds to $[g_i]_G$

- The Fischer matrix $F_i$,

$$F_i = \begin{pmatrix}
F_{i1} \\
F_{i2} \\
\vdots \\
F_{it}
\end{pmatrix}$$

Together with additional information required for their definition are presented as follows:
The Fischer Matrix $\mathcal{F}_i$ With Some Additional Information

<table>
<thead>
<tr>
<th>$g_i$</th>
<th>$\mathcal{F}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$g_{i1}$</td>
</tr>
<tr>
<td>$</td>
<td>C_G^{-1}(g_{ij})</td>
</tr>
<tr>
<td>$(k, m)$</td>
<td>$CH_k(g_{ikm})$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$g_i$</th>
<th>$\mathcal{F}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$g_{i1}$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(1,1)}$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(2,1)}$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(2,2)}$</td>
</tr>
<tr>
<td></td>
<td>$\cdots$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(2, r_2)}$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(u, 1)}$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(u, 2)}$</td>
</tr>
<tr>
<td></td>
<td>$\cdots$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(u, r_u)}$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(t, 1)}$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(t, 2)}$</td>
</tr>
<tr>
<td></td>
<td>$\cdots$</td>
</tr>
<tr>
<td></td>
<td>$a_{i1}^{(t, r_t)}$</td>
</tr>
<tr>
<td></td>
<td>$m_{ij}$</td>
</tr>
</tbody>
</table>
Properties of Fischer Matrices

The Fischer matrices satisfy some interesting properties, which help in computations of their entries.

(i) \[ \sum_{k=1}^{t} c(g_{ik}) = c(g_i), \]

(ii) \( \mathcal{F}_i \) is non-singular for each \( i \),

(iii) \( a_{ij}^{(1,1)} = 1, \ \forall \ 1 \leq j \leq c(g_i) \),

(iv) \( a_{11}^{(k,m)} = [G : H_k] \theta_k(1_N), \ \forall \ (k, m) \in J_1 \),

(v) For each \( 1 \leq i \leq r \), the weights \( m_{ij} \) satisfy the relation \[ \sum_{j=1}^{c(g_i)} m_{ij} = |N|, \]
(vi) **Column Orthogonality Relation:**

\[
\sum_{(k,m) \in J_i} |C_{H_k}(g_{ikm})| a_{ij}(k,m) \overline{a_{ij}(k,m)} = \delta_{jj'} |C_G(g_{ij})|, 
\]

(vii) **Row Orthogonality Relation:**

\[
\sum_{j=1}^{c(g_i)} m_{ij} a_{ij}(k,m) \overline{a_{ij}(k',m')} = \delta_{(k,m)(k',m')} a_{i1}(k,m) |N|. 
\]
Example of the Fischer Matrices

- Corresponding to \([2_2](3^{1+2}:8):2\), the Fischer matrix of \(\overline{G} = 2^{1+6}:(3^{1+2}:8):2\) will have the form:

<table>
<thead>
<tr>
<th>(g_3 = 2_2)</th>
<th>(F_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(o(g_{3j}))</td>
<td>(g_{31})</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>(</td>
<td>C_{\overline{G}}(g_{3j})</td>
</tr>
<tr>
<td>192</td>
<td>192</td>
</tr>
<tr>
<td>((k, m))</td>
<td>(m_{3j})</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>12</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>12</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>4</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>12</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>4</td>
</tr>
</tbody>
</table>

| 8 | 8 | 16 | 48 | 48 |

- By [8] the identity Fischer matrix \(F_1\) of the non-split extension group \(\overline{G}_n = 2^{2n} \cdot Sp(2n, 2)\) for any \(n \in \mathbb{N}^{\geq 2}\) will have the form:

<table>
<thead>
<tr>
<th>(g_1 = 1Sp(2n, 2))</th>
<th>(F_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(o(g_{1j}))</td>
<td>(g_{11})</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(</td>
<td>C_{\overline{G}}(g_{1j})</td>
</tr>
<tr>
<td>((k, m))</td>
<td>(</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>(\overline{G}_n)</td>
</tr>
<tr>
<td>((2, 1))</td>
<td>(\overline{G}_n/2^{2n} - 1)</td>
</tr>
<tr>
<td>(m_{1j})</td>
<td>(</td>
</tr>
</tbody>
</table>
Professor J. Moori has a significant contribution to this domain. Indeed he developed the coset analysis technique in his PhD thesis [15] and in [16].

Then together with his MSc and PhD students, they enriched this area of research by applying the coset analysis and Clifford-Fischer theory to many various split and non-split group extensions in a considerable number of publications. For example, but not limited to, one can refer to [1], [4, 5, 6, 7, 8, 9, 10], [17, 18], [20, 21, 22, 23], [25] or [26].

Barraclough produced an interesting PhD thesis [2], which contained a chapter on the method of Clifford-Fischer theory. He used this method to find the character table of any group of the form $2^2 \cdot G:2$ for any finite group $G$.

Also in 2007, H. Pahlings [24] calculated the Fischer matrices and the character table of the non-split extension $2^{1+22} \cdot C_{02}$, which is the second largest maximal subgroup of the Baby Monster group $\mathbb{B}$.

Then in 2010, H. Pahlings together with his student K. Lux published an interesting book [14] containing a full chapter on Clifford-Fischer theory that includes several examples on the application of the method.
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