New progress on factorized groups and subgroup permutability

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Factorized groups:

- All groups considered will be finite.

Factorized groups: $A$ and $B$ subgroups of a group $G$

$$G = AB$$

- How the structure of the factors $A$ and $B$ affects the structure of the whole group $G$?
- How the structure of the group $G$ affects the structure of $A$ and $B$?
Natural approach: **Classes of groups**

A class of groups is a collection $\mathcal{F}$ of groups with the property that if $G \in \mathcal{F}$ and $G \cong H$, then $H \in \mathcal{F}$.
Factorized groups

Natural approach: Classes of groups

A class of groups is a collection $\mathcal{F}$ of groups with the property that if $G \in \mathcal{F}$ and $G \cong H$, then $H \in \mathcal{F}$

Question

Let $\mathcal{F}$ be a class of groups and $G = AB$ a factorized group:

- $A, B \in \mathcal{F} \implies G \in \mathcal{F}$?
- $G \in \mathcal{F} \implies A, B \in \mathcal{F}$?
Definitions

- A **formation** is a class $\mathcal{F}$ of groups with the following properties:
  - Every homomorphic image of an $\mathcal{F}$-group is an $\mathcal{F}$-group.
  - If $G/M$ and $G/N \in \mathcal{F}$, then $G/(M \cap N) \in \mathcal{F}$

- $\mathcal{F}$ a formation: the $\mathcal{F}$-residual $G^\mathcal{F}$ of $G$ is the smallest normal subgroup of $G$ such that $G/G^\mathcal{F} \in \mathcal{F}$

- The formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$, then $G \in \mathcal{F}$.
Starting point

\[ G = AB : \ A, B \in \mathcal{U}, \ A, B \trianglelefteq G \iff G \in \mathcal{U} \]

Example

\[ Q = \langle x, y \rangle \cong Q_8, \quad V = \langle a, b \rangle \cong C_5 \times C_5 \]

\[ G = [V]Q \text{ the semidirect product of } V \text{ with } Q \]

\[ G = AB \text{ with } A = V \langle x \rangle \text{ and } B = V \langle y \rangle \]

\[ A, B \in \mathcal{U}, \ A, B \trianglelefteq G, \ G \notin \mathcal{U} \]
Starting point

\[ G = AB : \ A, B \in \mathcal{U}, \ A, B \trianglelefteq G \nRightarrow G \in \mathcal{U} \]

Example

\[ Q = \langle x, y \rangle \cong Q_8, \quad V = \langle a, b \rangle \cong C_5 \times C_5 \]
\[ G = [V]Q \text{ the semidirect product of } V \text{ with } Q \]
\[ G = AB \text{ with } A = V\langle x \rangle \text{ and } B = V\langle y \rangle \]
\[ A, B \in \mathcal{U}, \ A, B \trianglelefteq G, \ G \notin \mathcal{U} \]

\[ G = AB : \ A, B \in \mathcal{U}, \ A, B \trianglelefteq G + \text{additional conditions} \implies G \in \mathcal{U} \]

- (Baer, 57) \( G' \in \mathcal{N} \)
- (Friesen, 71) \( (|G : A|, |G : B|) = 1 \)
Permutability properties

If $G = AB$ is a central product of the subgroups $A$ and $B$, then:

$$A, B \in \mathcal{U} \implies G \in \mathcal{U}$$

More generally, if $\mathcal{F}$ is any formation:

$$A, B \in \mathcal{F} \implies G \in \mathcal{F}$$

(In particular, this holds when $G = A \times B$ is a direct product.)
Permutability properties

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(In particular, this holds when $G = A \times B$ is a direct product.)

Let $G = AB$ a factorized group:

$$A, B \in \mathcal{U} \ ( \text{or } \mathcal{F}) \quad \text{permutability properties} \quad \implies G \in \mathcal{U} \ ( \text{or } \mathcal{F})$$
Permutability properties

Total permutability

Definition

Let $G$ be a group and let $A$ and $B$ be subgroups of $G$. It is said that $A$ and $B$ are totally permutable if every subgroup of $A$ permutes with every subgroup of $B$.

Theorem

(Asaad, Shaalan, 89) If $G = AB$ is the product of the totally permutable subgroups $A$ and $B$, then

$$A, B \in \mathcal{U} \implies G \in \mathcal{U}$$
Total permutability and formations

(Maier,92; Carocca,96; Ballester-Bolinches, Pedraza-Aguilera, Pérez-Ramos, 96-98) Let $\mathcal{F}$ be a formation such that $\mathcal{U} \subseteq \mathcal{F}$. Let the group $G = G_1 G_2 \cdots G_r$ be a product of pairwise totally permutable subgroups $G_1, G_2, \ldots, G_r, r \geq 2$. Then:

**Theorem**

- If $G_i \in \mathcal{F}$ $\forall i \in \{1, \ldots, r\}$, then $G \in \mathcal{F}$.
- Assume in addition that $\mathcal{F}$ is either saturated or $\mathcal{F} \subseteq S$. If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}, \forall i \in \{1, \ldots, r\}$.

**Corollary**

- If $\mathcal{F}$ is either saturated or $\mathcal{F} \subseteq S$, then: $G^{\mathcal{F}} = G_1^{\mathcal{F}} G_2^{\mathcal{F}} \cdots G_r^{\mathcal{F}}$. 
Conditional permutability

Definitions

(Qian, Zhu, 98) (Guo, Shum, Skiba, 05) Let $G$ be a group and let $A$ and $B$ be subgroups of $G$.

- $A$ and $B$ are **conditionally permutable** in $G$ (c-permutable), if $AB^g = B^gA$ for some $g \in G$.
- $A$ and $B$ are **totally c-permutable** if every subgroup of $A$ is c-permutable in $G$ with every subgroup of $B$.

Example

Let $X$ and $Y$ be two 2-Sylow subgroups of $S_3$. Then $X$ permutes with $Y^g$ for some $g \in S_3$, but $X$ does not permute with $Y$. 
Total $c$-permutability and supersolubility

**Theorem**

(Arroyo-Jordá, AJ, Martínez-Pastor, Pérez-Ramos, 10) Let $G = AB$ be the product of the totally $c$-permutable subgroups $A$ and $B$. Then:

$$G^U = A^U B^U$$
Total c-permutability and supersolubility

**Theorem**

(Arroyo-Jordá, AJ, Martínez-Pastor, Pérez-Ramos, 10) Let $G = AB$ be the product of the totally c-permutable subgroups $A$ and $B$. Then:

$$G^\mathcal{U} = A^\mathcal{U} B^\mathcal{U}$$

In particular, $A, B \in \mathcal{U} \iff G \in \mathcal{U}$
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In particular, $A, B \in \mathcal{U} \iff G \in \mathcal{U}$

**Corollary**

(AJ, AJ, MP, PR, 10) Let $G = AB$ be the product of the totally c-permutable subgroups $A$ and $B$ and let $p$ be a prime. If $A, B$ are $p$-supersoluble, then $G$ is $p$-supersoluble.
Question

Are saturated formations $F$ (of soluble groups) containing $U$ closed under taking products of totally $c$-permutable subgroups?

Example

Take $G = S_4 = AB$, $A = A_4$ and $B \cong C_2$ generated by a transposition. Then $A$ and $B$ are totally $c$-permutable in $G$.

Let $F = N^2$, the saturated formation of metanilpotent groups. Notice $U \subseteq N^2 \subseteq S$. Then:

$$A, B \in F \text{ but } G \notin F.$$ 

In particular, $G^F \neq A^F B^F$. 
Conditional permutability

Remark

c-permutability fails to satisfy the property of persistence in intermediate subgroups.

Example

Let $G = S_4$ and let $Y \cong C_2$ generated by a transposition.

Let $V$ be the normal subgroup of $G$ of order 4 and $X$ a subgroup of $V$ of order 2, $X \neq Z(VY)$. Then

- $X$ and $Y$ are c-permutable in $G$
- $X$ and $Y$ are not c-permutable in $\langle X, Y \rangle$. 
Complete c-permutability

Definitions

(Guo, Shum, Skiba, 05) Let $G$ be a group and let $A$ and $B$ be subgroups of $G$.

- $A$ and $B$ are **completely c-permutable** in $G$ (cc-permutable), if $AB^g = B^gA$ for some $g \in \langle A, B \rangle$.

- $A$ and $B$ are **totally completely c-permutable** (tcc-permutable) if every subgroup of $A$ is completely c-permutable in $G$ with every subgroup of $B$.

| Totally permutable | $\implies$ | Totally completely c-permutable | $\implies$ | Totally c-permutable | $\nRightarrow$ |
Complete c-permutability and supersolubility

\[ G = AB, \ A, B \text{ totally c-permutable}, \ G^U = A^U B^U \]

**Corollary**

(Guo, Shum, Skiba, 06)

- Let \( G = AB \) be a product of the tcc-permutable subgroups \( A \) and \( B \). If \( A, B \in \mathcal{U} \), then \( G \in \mathcal{U} \).
Complete c-permutability and supersolubility

\[ G = AB, \ A, B \text{ totally c-permutable, } G^\mathcal{U} = A^\mathcal{U} B^\mathcal{U} \]

**Corollary**

(Guo, Shum, Skiba, 06)

- Let \( G = AB \) be a product of the tcc-permutable subgroups \( A \) and \( B \). If \( A, B \in \mathcal{U} \), then \( G \in \mathcal{U} \).

- Let \( G = AB \) be the product of the tcc-permutable subgroups \( A \) and \( B \) and let \( p \) be a prime. If \( A, B \) are \( p \)-supersoluble, then \( G \) is \( p \)-supersoluble.
**Question**

Are saturated formations $\mathcal{F}$ (of soluble groups) containing $\mathcal{U}$ closed under taking products of totally completely c-permutable subgroups?

**Theorem**


Let $\mathcal{F}$ be a saturated formation such that $\mathcal{U} \subseteq \mathcal{F} \subseteq S$.

Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that $G_i$ and $G_j$ are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then:

- If $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$, then $G \in \mathcal{F}$.
- If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$. 
Total complete c-permutability and saturated formations

**Corollary**


Let $F$ be a saturated formation such that $U \subseteq F \subseteq S$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that $G_i$ and $G_j$ are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then:

- $G_i^F \leq G$ for all $i = 1, \ldots, r$.
- $G^F = G_1^F \cdots G_r^F$. 
Total complete c-permutability and saturated formations

**Question**

Is it possible to extend the above results on products of tcc-permutable subgroups to either non-saturated formations or saturated formations of non-soluble groups $\mathcal{F}$ such that $\mathcal{U} \subseteq \mathcal{F}$?
Total complete c-permutability and saturated formations

**Question**

Is it possible to extend the above results on products of tcc-permutable subgroups to either non-saturated formations or saturated formations of non-soluble groups $F$ such that $U \subseteq F$?

- We need a better knowledge of structural properties of products of tcc-permutable groups.
Lemma

(AJ, AJ, PR, 11) If $1 \neq G = AB$ is the product of tcc-permutable subgroups $A$ and $B$, then there exists $1 \neq N \trianglelefteq G$ such that either $N \leq A$ or $N \leq B$.

Corollary

(AJ, AJ, PR, 11) Let the group $1 \neq G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that $G_i$ and $G_j$ are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$.

Then there exists $1 \neq N \trianglelefteq G$ such that $N \leq G_i$ for some $i \in \{1, \ldots, r\}$. 
Subnormal subgroups

**Proposition**

(AJ,AJ,MP,PR,13) Let the group $G = AB$ be the product of tcc-permutable subgroups $A$ and $B$. Then

$$A' 	rianglelefteq 	rianglelefteq G \quad \text{and} \quad B' 	rianglelefteq 	rianglelefteq G.$$ 

**Corollary**

(AJ,AJ,MP,PR,13) Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that $G_i$ and $G_j$ are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then:

$$G'_i \trianglelefteq \trianglelefteq G, \quad \text{for all } i \in \{1, \ldots, r\}.$$
Subnormal subgroups

**Proposition**

(Maier, 92) If \( G = AB \) is the product of totally permutable subgroups \( A \) and \( B \), then \( A \cap B \leq F(G) \), that is, \( A \cap B \) is a subnormal nilpotent subgroup of \( G \).

**Example**

The above property is not true for products of tcc-permutable subgroups.

- Let \( G = S_3 = AB \) with the trivial factorization \( A = S_3 \) and \( B \) a 2-Sylow subgroup of \( G \). This is a product of tcc-permutable subgroups, but: \( A \cap B = B \) is not a subnormal subgroup of \( G \).

- Let \( G = S_3 = AB \) with the trivial factorization \( A = B = S_3 \). This is a product of tcc-permutable subgroups, but: \( A \cap B = S_3 \notin \mathcal{N} \).
Nilpotent residuals

**Theorem**

(Beidleman, Heineken, 99) Let \( G = AB \) be a product of the totally permutable subgroups \( A \) and \( B \). Then:

\[
[A^N, B] = 1 \quad \text{and} \quad [B^N, A] = 1.
\]

**Example**

Let \( V = \langle a, b \rangle \cong C_5 \times C_5 \) and \( C_6 \cong C = \langle \alpha, \beta \rangle \leq \text{Aut}(V) \) given by:

\[
a^\alpha = a^{-1}, \quad b^\alpha = b^{-1}; \quad a^\beta = b, \quad b^\beta = a^{-1}b^{-1}
\]

Let \( G = [V]C \) be the corresponding semidirect product. Then \( G = AB \) is the product of the tcc-permutable subgroups \( A = \langle \alpha \rangle \) and \( B = V\langle \beta \rangle \). Notice that \( A \in \mathcal{U} \), but

\[
B^N = B^\mathcal{U} = V \quad \text{does not centralize} \ A.
\]
Nilpotent residuals

**Theorem**

(AJ,AJ,MP,PR,13) Let the group $G = AB$ be the product of tcc-permutable subgroups $A$ and $B$. Then

$$A^N \trianglelefteq G \text{ and } B^N \trianglelefteq G.$$ 

**Corollary**

(AJ,AJ,MP,PR,13) Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that $G_i$ and $G_j$ are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then

$$G_i^N \trianglelefteq G, \text{ for all } i \in \{1, \ldots, r\}.$$
\[U\)-hypercentre

**Theorem**

(Hauck, PR, MP, 03), (Gállego, Hauck, PR, 08) *Let* \( G = AB \) *be a product of the totally permutable subgroups* \( A \) *and* \( B \). *Then:*

\[ [A, B] \leq Z_U(G) \]

*or, equivalently,* \( G/Z_U(G) = AZ_U(G)/Z_U(G) \times BZ_U(G)/Z_U(G) \).

**Example**

Let \( G = [V]C = AB \) *the product of the tcc-permutable subgroups* \( A = \langle \alpha \rangle \) *and* \( B = V\langle \beta \rangle \) *under the action* \( a^\alpha = a^{-1}, b^\alpha = b^{-1}; a^\beta = b, b^\beta = a^{-1}b^{-1} \). *Notice that:*

\[ Z_U(G) = 1 \) *and* \( G \) *is not a direct product of* \( A \) *and* \( B \).
Main theorem

**Theorem**

(AJ,AJ,MP,PR, 13)

*Let the group $G = AB$ be the product of tcc-permutable subgroups $A$ and $B$. Then:*

$$[A, B] \leq F(G).$$

*For the proof we have used the CFSG.*
Consequences of the main theorem

Corollary

(AJ,AJ,MP,PR, 13) Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$, and $G_i \neq 1$ for all $i = 1, \ldots, r$. Assume that $G_i$ and $G_j$ are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Let $N$ be a minimal normal subgroup of $G$. Then:

1. If $N$ is non-abelian, then there exists a unique $i \in \{1, \ldots, r\}$ such that $N \leq G_i$. Moreover, $G_j$ centralizes $N$ and $N \cap G_j = 1$, for all $j \in \{1, \ldots, r\}$, $j \neq i$. 


Consequences of the main theorem

Corollary

(AJ,AJ,MP,PR, 13) Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$, and $G_i \neq 1$ for all $i = 1, \ldots, r$. Assume that $G_i$ and $G_j$ are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Let $N$ be a minimal normal subgroup of $G$. Then:

1. If $N$ is non-abelian, then there exists a unique $i \in \{1, \ldots, r\}$ such that $N \leq G_i$. Moreover, $G_j$ centralizes $N$ and $N \cap G_j = 1$, for all $j \in \{1, \ldots, r\}$, $j \neq i$.

2. If $G$ is a monolithic primitive group, then the unique minimal normal subgroup $N$ is abelian.
Consequences of the main theorem

**Corollary**

(AJ,AJ,MP,PR, 13) Let the group $G = AB$ be the tcc-permutable product of the subgroups $A$ and $B$. Then:

- If $A$ is a normal subgroup of $G$, then $B$ acts $u$-hypercentrally on $A$ by conjugation. In particular, $B^U$ centralizes $A$.
Total complete c-permutability and formations

**Theorem**

(AJ, AJ, MP, PR, 13) Let $\mathcal{F}$ be a **saturated** formation such that $\mathcal{U} \subseteq \mathcal{F}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that $G_i$ and $G_j$ are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then:

- If $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$, then $G \in \mathcal{F}$.
- If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$. 
Theorem

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- If $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$, then $G \in \mathcal{F}$.
- If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$.

Corollary

Under the same hypotheses:

- $G_i^\mathcal{F} \trianglelefteq G$ for all $i = 1, \ldots, r$.
- $G^\mathcal{F} = G_1^\mathcal{F} \cdots G_r^\mathcal{F}$.
Necessity of saturation

Example

Define the mapping $f : \mathbb{P} \longrightarrow \{ \text{classes of groups} \}$ by setting

$$f(p) = \begin{cases} (1, C_2, C_3, C_4) & \text{if } p = 5 \\ (G \in A : \exp(G) | p - 1) & \text{if } p \neq 5 \end{cases}$$

Let $\mathcal{F} = \{ G \in S \mid H/K \text{ chief factor of } G \Rightarrow \text{Aut}_G(H/K) \in f(p) \ \forall p \in \sigma(H/K) \}$. $\mathcal{F}$ is a formation of soluble groups such that $\mathcal{U} \subseteq \mathcal{F}$.

Let again $G = [V]C = AB$ be the product of the tcc-permutable subgroups $A = \langle \alpha \rangle$ and $B = V\langle \beta \rangle$ (under the action $a^\alpha = a^{-1}$, $b^\alpha = b^{-1}$; $a^\beta = b$, $b^\beta = a^{-1}b^{-1}$). Then:

- $A, B \in \mathcal{F}$, but $G \not\in \mathcal{F}$, since $G/C_G(V) \cong C_3 \times C_2 \not\in f(5)$. 
Necessity of saturation

Example

Define the mapping \( f : \mathbb{P} \rightarrow \{ \text{classes of groups} \} \) by setting

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f(p) = \begin{cases} 
(1, C_2, C_3, C_4) & \text{if } p = 5 \\
(G \in A : \exp(G) \mid p - 1) & \text{if } p \neq 5
\end{cases}
\]

Let \( \mathcal{F} = (G \in S \mid H/K \text{ chief factor of } G \Rightarrow \text{Aut}_G(H/K) \in f(p) \ \forall p \in \sigma(H/K)) \).

\( \mathcal{F} \) is a formation of soluble groups such that \( \mathcal{U} \subseteq \mathcal{F} \).

Let again \( G = [V]C = AB \) be the product of the tcc-permutable subgroups \( A = \langle \alpha \rangle \) and \( B = V \langle \beta \rangle \) (under the action \( a^\alpha = a^{-1}, b^\alpha = b^{-1}; a^\beta = b, b^\beta = a^{-1}b^{-1} \)). Then:

- \( A, B \in \mathcal{F} \), but \( G \not\in \mathcal{F} \), since \( G/C_G(V) \cong C_3 \times C_2 \not\in f(5) \).

Modifying the construction of the formation \( \mathcal{F} \) by setting \( f(5) = (1, C_2, C_4, C_6) \):

- \( G, A \in \mathcal{F} \), but \( B \not\in \mathcal{F} \), since \( B/C_B(V) \cong C_3 \not\in f(5) \).
References


THANK YOU FOR YOUR ATTENTION!