Hausdorff dimension in groups acting on trees

Amaia Zugadi-Reizabal

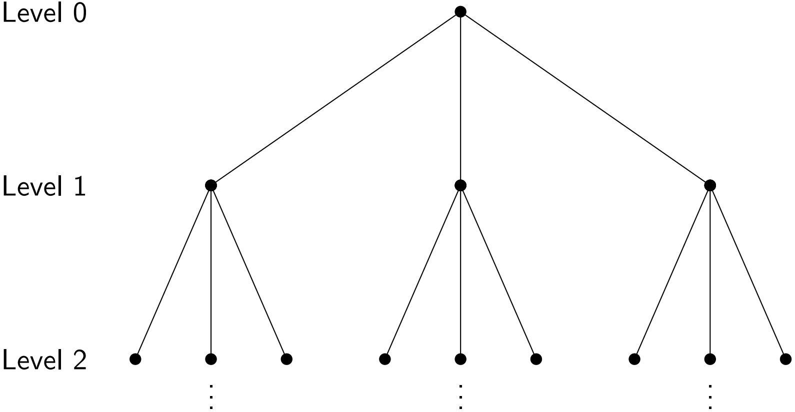
University of the Basque Country, Bilbao

Group St Andrews,
Bath, August 4th 2009

Contents

Contents

An example: Regular rooted ternary tree



Automorphisms of rooted trees

Definition

An *automorphism* of a rooted tree, $\mathcal{T}$, is a bijection of the vertices that preserves incidence.
We denote the group of automorphisms of $\mathcal{T}$ by $\operatorname{Aut}\mathcal{T}$.

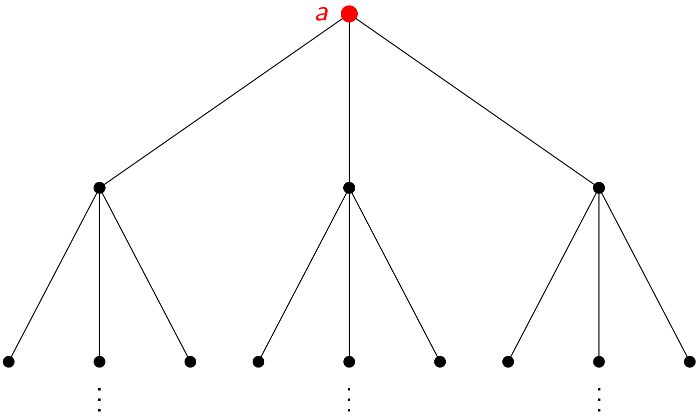
First properties

Let $f \in \operatorname{Aut}\mathcal{T}$, then

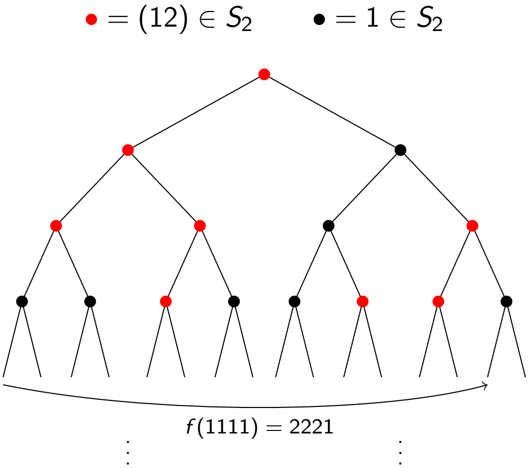
- f fixes the root $\emptyset$.
- f preserves the levels (f preserves the distance and the n-th level is the sphere of radius n and centered at the root).
- The image of a vertex under f determines the images of all the 'predecessors'.

An example (through its *portrait*)

$$a = (1 \dots p) \in \operatorname{Aut} \mathcal{T}$$



Another example (through its portrait)



The structure of $\mathcal{A} = \operatorname{Aut} \mathcal{T}(p)$

Definition

The subgroup $\operatorname{Stab}(n)$ of $\mathcal{A}$ consisting of the automorphisms that fix the nth level is called the *nth level stabilizer*.

Remark

$\operatorname{Stab}(n)$ is normal in $\mathcal{A}$ and $\mathcal{A}/\operatorname{Stab}(n) \cong \operatorname{Aut} \mathcal{T}_n$.

Theorem

If $\mathcal{A} = \operatorname{Aut} \mathcal{T}(p)$, then

$$\mathcal{A} = \varprojlim_{n \in \mathbb{N}} \mathcal{A}/\operatorname{Stab}(n) \cong ((S_p \wr S_p) \wr S_p) \wr \ldots$$

is a profinite group.

Well-known groups

Grigorchuk group ($p = 2$)

- The first example of group of intermediate word growth.
- General Burnside Problem: 3-generated by elements of order 2, infinite and periodic.

Gupta-Sidki group ($p > 2$)

General Burnside Problem: 2-generated by elements of order p, infinite and periodic.

Contents

What is the Hausdorff dimension?

Hausdorff dimension is a way of measuring the relative size of a subgroup in the whole group.
It was originally defined for metric spaces and it is a sharp tool to detect the 'fractalness' of sets.

Hausdorff dimension in profinite groups

Suppose G is a countably based profinite group i.e. there exists a descending chain $\{G_n\}_{n\in\mathbb{N}}$ of open normal subgroups which form a base of neighbourhoods of the identity (if G is (topologically) finitely generated, then it is countably based).
Let H be a closed subgroup of G. Then, as **Abercrombie** and **Barnea-Shalev** proved, the Hausdorff dimension of H in G is:

$$\dim_G H = \liminf_{n\to\infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|}.$$

The spectrum of G

Definition

$\mathrm{Spec}(G) = \{\dim_G H : H \leq_c G\} \subseteq [0,1]$ is the *spectrum* of G.

It is useful if we want to measure the 'complexity' of the subgroup structure of G.

Theorem (Barnea and Shalev)

Let G be a *p-adic analytic pro-p group* and let d denote the dimension of G as a Lie group, then

$$\mathrm{Spec}(G) \subseteq \left\{0, \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}, 1\right\}$$

is finite and contains just rational numbers.

An important result

Let $c \in S_p$ be a p-cycle and let us consider the subgroup of $\mathcal{A}$ obtained considering all automorphisms that have just powers of c on their portraits. Then that subgroup is a Sylow pro-p subgroup Γ of $\mathcal{A}$ containing c.

Theorem (Abért-Virág)

For every $\lambda \in [0,1]$, there exists $H \leq_c \Gamma$ (top.) finitely generated by 3 elements such that $\dim_\Gamma H = \lambda$.
Therefore, $\mathrm{Spec}(\Gamma) = [0,1]$.

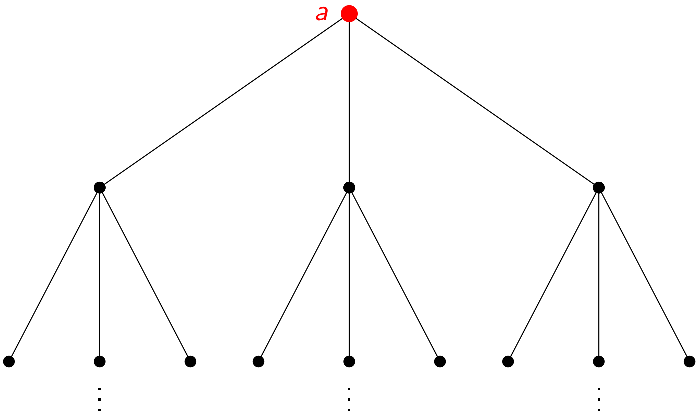
Remarks

The methods used to prove this result are probabilistic and they do not give specific examples.
Siegenthaler shows examples of 3-generated *spinal groups* of transcendental Hausdorff dimension for the binary tree.

Contents

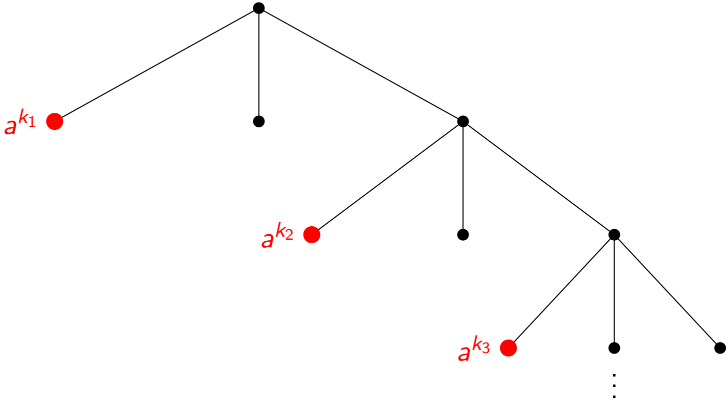
The generator $a = (1 \dots p)$

$$a = (1 \dots p) \in \operatorname{Aut} \mathcal{T}$$



Spinal generators

Fixed a sequence of powers $\mathbf{k} = \{k_i\}_{i=1}^{\infty}$, then let us define the spinal generator corresponding to $\mathbf{k}$ as:



Hausdorff dimension of spinal groups ($p = 2$)

Siegenthaler gives an explicit formula to compute the Hausdorff dimension for the spinal groups of the binary tree.

Theorem (Siegenthaler)

The spinal spectrum for $p = 2$ contains several copies of a Cantor set.
Therefore, it contains transcendental elements.

Hausdorff dimension of spinal groups ($p > 2$)

The situation is quite different in this case:

Theorem (Fernández-Alcober, Z-R)

If $p > 2$, then

$$\dim_\Gamma \overline{G} = (p-1) \liminf_{m \to \infty} \left(\frac{1}{p} + \frac{1}{p^{r_1}} + \ldots + \frac{1}{p^{r_{n-1}}} \right),$$

where $r_j = r_j(m)$ and n are natural numbers that depend on G.

Spinal spectrum ($p > 2$)

Theorem (Fernández-Alcober, Z-R)

The spinal spectrum for $p > 2$ is

$$\left\{ \frac{x_1}{p} + \frac{x_2}{p^2} + \ldots + \frac{x_n}{p^n} : x_1 = p-1,\ x_i = 0 \text{ or } p-1 \text{ and } n \in \mathbb{N} \right\}.$$

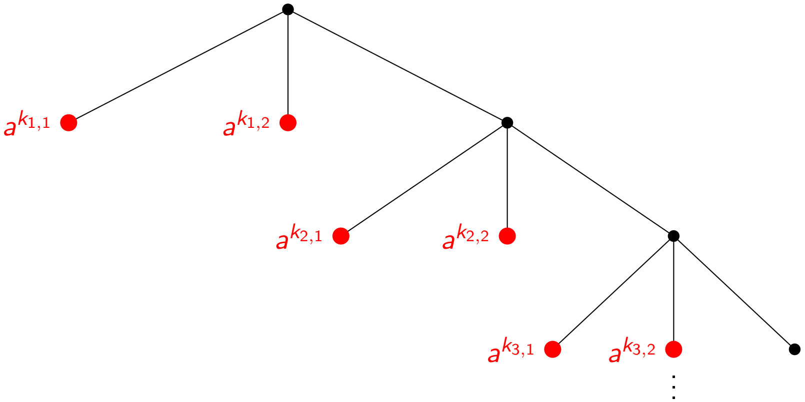
We give an algorithm that, given $\lambda \in [0,1]$ of the appropriate type, provides a spinal group whose Hausdorff dimension is λ.

Remarks

All these numbers are rational and $\geq \frac{p-1}{p}$.

Next step: consider *multi-edge* spinal groups

Fixed a sequence of tuples of powers $\mathbf{k} = \{(k_{i,1}, \ldots, k_{i,p-1})\}_{i=1}^{\infty}$, then let us define the multi-edge spinal generator corresponding to $\mathbf{k}$ as:



Constant versus Variable

Theorem (Fernández-Alcober, Z-R)

Let G be a two-generated spinal group corresponding to a constant sequence $\mathbf{k} = \{(k_1, \ldots, k_{p-1})\}_{i=1}^{\infty}$. Then

$$\dim_{\Gamma} \overline{G} = \frac{(p-1)t}{p^2} - \frac{r}{p^2} - \frac{s}{p^2(p-1)},$$

where t, r and s are natural numbers that depend on $\mathbf{k}$.
In particular, all two-generated constant multi-edge spinal groups have rational Hausdorff dimension.

Theorem (Fernández-Alcober, Z-R)

The spectrum of two-generated variable multi-edge spinal groups contains transcendental Hausdorff dimensions.

Thank you!
Eskerrik asko!

:-)