$2^6{:}A_8$ as an inertia factor group of $2^8{:}O_8^+(2)$

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Introduction

- $2^8{:}O_8^+(2)$ has three inertia factor groups namely $O_8^+(2)$, $Sp(6,2)$ and $2^6{:}A_8$. Now $2^6{:}A_8$ is a maximal subgroup of $O_8^+(2)$ of index 135 and order 1 290 240. Since $2^6{:}A_8$ is an inertia group, it plays an essential role in the construction of the character table of $2^8{:}O_8^+(2)$ as there is a block in the table that corresponds to $2^6{:}A_8$.
- A_8 acts on the 6 dimensional vector space 2^6. The action of A_8 on 2^6 is multiplication from the right of 2^6 by A_8, hence A_8 should be represented by 6×6 matrices.
- In this paper we look at two ways to construct $2^6{:}A_8$. In the first method we use combinotorics to construct A_8 and then with help from the Atlas we find the the inertia factors.
- For the second method we use GAP to construct A_8 inside $O_8^+(2)$ and again using GAP, we compute the inertia factors of A_8. The two groups constructed are isomorphic.

Definitions

- $\overline{G}$ is said to be a semi direct product of N by G if for $N, G \leq \overline{G}$ we have :
  (i) $\overline{G} = NG$
  (ii) $N \trianglelefteq \overline{G}$
  (iii) $N \cap G = \{1_G\}$
- An orthogonal group of degree n over a finite field $\mathbb{F}_q$ written $O_n^{\epsilon}(q)$ is the group of $n \times n$ orthogonal matrices, with entries from $\mathbb{F}_q$, with the group operation that of matrix multiplication.
- Let G be a group and $H \leq G$. Then for a character χ of H we define

$$I_G(\chi) = \{g \in N_G(H) | \chi^g = \chi\}$$

and we call $I_G(\chi)$ the *inertia group* of χ in G. If $H \trianglelefteq G$ then

$$I_G(\chi) = \{g \in G | \chi^g = \chi\}$$

The first method

- The group S_8 acts naturally on a module of dimension 8 by permuting the basis elements which generate the module. Let V be the 8-dimensional natural module of S_8 over $GF(2)$, where $V =< e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 >$, and $e_i^2 = 1$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ where we regard V as a multiplicative elementary abelian 2-group of order 2^8.

Theorem

Let V be the natural module of S_8 over $GF(2)$. Then there exist S_8 submodules M_1 and M_2 of V such that $V \supset M_2 \supset M_1 \supset 0$ and that

$$dim(M_2) = 7 \qquad \text{and} \qquad dim(M_1) = 1. \ \square$$

- **Note** Since S_8 is 8 transitive, A_8 is six transitive on $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

Theorem

Let $W = M_2/M_1$, then $dim(W) = 6$. Also W is a G - invariant submodule of V where $G = S_8$ or A_8. $\square$

- Let

  $$W =< e_1e_2M_1, e_1e_3M_1, e_1e_4M_1, e_1e_5M_1, e_1e_6M_1, e_1e_7M_1 >$$

  Also let $\gamma_1 = e_1e_2M_1, \gamma_2 = e_1e_3M_1, \gamma_3 = e_1e_4M_1, \gamma_4 = e_1e_5M_1, \gamma_5 = e_1e_6M_1, \gamma_6 = e_1e_7M_1$. Also, if $\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7)$ and $\beta = (6\ 7\ 8)$ then $A_8 =< \alpha, \beta >$.
- We get that :
  $\alpha : \gamma_1 \to \gamma_1\gamma_2,\ \gamma_2 \to \gamma_1\gamma_3,\ \gamma_3 \to \gamma_1\gamma_4,\ \gamma_4 \to \gamma_1\gamma_5,\ \gamma_5 \to \gamma_1\gamma_6,\ \text{and}\quad \gamma_6 \to \gamma_1$
- We also get :
  $\beta : \gamma_1 \to \gamma_1,\ \gamma_2 \to \gamma_2,\ \gamma_3 \to \gamma_3,\ \gamma_4 \to \gamma_4,\ \gamma_5 \to \gamma_6,\ \gamma_6 \to \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$

Generators

- We give two examples for the action of α. Under the action of α we have $\gamma_2 = e_1e_3M_1 \rightarrow e_2e_4M_1 = e_1e_2e_1e_4M_1 = \gamma_1\gamma_3$ and also have $\gamma_6 = e_1e_7M_1 \rightarrow e_2e_1M_1 = \gamma_1$.
- We also give two for the action of β. Under the action of β we have $\gamma_5 = e_1e_6M_1 \rightarrow e_1e_7M_1 = \gamma_6$ and $\gamma_6 = e_1e_7M_1 \rightarrow e_1e_8M_1 = e_2e_3e_4e_5e_6e_7M_1 = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$
- Hence $\alpha and \beta$ are represented by the following matrices respectively where $o(\alpha) = 7$ and $o(\beta) = 3$:

$$\alpha = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Inertia Factors

- A_8 acts irreducibly on W and we get three orbits, namely Δ_0, Δ_1 and Δ_2 where $|\Delta_0| = 1, |\Delta_1| = 28$ and $|\Delta_2| = 35$
- These have corresponding point stabilizers K_1, K_2 and K_3 of indices $1, 28$ and 35 respectively. From the ATLAS we get up to isomorphism and conjugacy that $K_1 \cong A_8,\ \ K_2 \cong S_6$ and $K_3 \cong 2^4 : (S_3 \times S_3)$

The Second Method

- Extensive use of GAP is followed here. We first construct $O_8^+(2)$ inside the general orthogonal group $GO_8^+(2)$:
  (i) We first construct the general orthogonal group $GO_8^+(2)$.
  (ii) Get maximal normal subgroups of $GO_8^+(2)$.
  (ii) A group of 8x8 matrices of size 174182400 in characteristic 2
  (iv) $Gr := O_8^+(2)$
- We then construct $2^6{:}A_8$ inside $O_8^+(2)$:
  (i) We construct an eight dimensional row vector space V over $GF(2)$.
  (ii) Let Gr act on V .
  (iii) Get the orbit lengths.
  (iv) Take the representative of the orbit of length 135.
  (v) Find the stabilizer of the representative in Gr
  (v) A group of 8x8 matrices of size 1290240 .
  (vi) $gr2 := 2^6{:}A_8$.

Generators of 2^6

- We compute the generators of 2^6 again using GAP :
  (i) Find the normal subgroups of *gr*2.
  (ii) Take the proper normal subgroup.
  (ii) A matrix group of size 64 with 6 generators.

The Generators of 2^6

- We give the generators of 2^6 in the table below :

<u>The Generators of 2^6</u>

$gen[1] = \begin{bmatrix} 1&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$	$gen[4] = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&1&0&0 \\ 1&0&0&1&0&1&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$
$gen[2] = \begin{bmatrix} 1&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&1&0&1 \\ 1&1&0&0&1&1&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$	$gen[5] = \begin{bmatrix} 1&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&1&0&0 \\ 0&1&0&1&0&1&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$
$gen[3] = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&1 \\ 1&0&0&0&1&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$	$gen[6] = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&1 \\ 0&0&0&1&0&0&0&1 \\ 0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&1&0&1 \\ 0&0&1&1&1&1&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$

Second Version of A_8

- We now construct A_8 from our $2^6{:}A_8$:
  (i) Find the generators of $gr2 = 2^6{:}A_8$.
  (ii) Take the two generators of order 4 each and form a group.
  (iii) A matrix group with 2 generators of size 20160.
  (iv) grup1:=A_8

8×8 Generators

The Generators of A_8

$$a = \begin{bmatrix} 1&0&0&0&0&1&0&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&0&1&0&0\\ 0&0&0&1&0&0&0&0\\ 1&1&0&0&1&1&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \qquad a^{-1} = \begin{bmatrix} 1&0&0&0&0&1&0&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&1&0&1&1&1&0&1\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 0&0&1&1&1&1&1&0\\ 1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&0&1\\ 1&1&0&0&0&1&1&1\\ 1&1&1&1&0&1&0&1\\ 0&1&0&0&0&0&1&0\\ 1&1&1&1&0&0&0&0 \end{bmatrix} \qquad b^{-1} = \begin{bmatrix} 0&0&0&0&1&1&1&1\\ 0&0&1&1&1&1&1&0\\ 1&0&0&1&1&0&1&0\\ 1&0&1&0&1&0&1&0\\ 1&1&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&0&0\\ 1&0&0&0&0&1&0&1 \end{bmatrix}$$

- Letting $gen[i] = \gamma_i, \quad i = 1, \cdots, 6$, the conjugate $a\gamma_i a^{-1}$ we get $a\gamma_i a^{-1} = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_k}$ for some $j_r = 1, \cdots, 6$. If we denote this as $\gamma_i \to \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_k}$.We then get :
  $\gamma_1 \to \gamma_1$, $\gamma_2 \to \gamma_2\gamma_5$, $\gamma_3 \to \gamma_2\gamma_4\gamma_5\gamma_6$, $\gamma_4 \to \gamma_4$, $\gamma_5 \to \gamma_1\gamma_4\gamma_5$, $\gamma_6 \to \gamma_1\gamma_2\gamma_3\gamma_4$.

6×6 Generators

- Similarly with b we get
  $\gamma_1 \to \gamma_1\gamma_2\gamma_3\gamma_4\gamma_6$, $\gamma_2 \to \gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$, $\gamma_3 \to \gamma_1\gamma_2\gamma_3\gamma_5\gamma_6$, $\gamma_4 \to \gamma_1\gamma_3\gamma_4$, $\gamma_5 \to \gamma_4$, $\gamma_6 \to \gamma_2\gamma_3\gamma_4$
- We then get our 6×6 generators of A_8 :

The Generators of A_8

$grup2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$	$grup3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$

- grup:=Group(grup2,grup3);

Second set of inertia factors

We now find the inertia factors of A_8:

(i) Construct a row vector space U of dimension 6 over $GF(2)$.

(ii) Find orbits when *grup* acts on U.

(iii) Find orbitlengths .

(iv) Take representative of orbit of length 28 and find its stabilizer in *grup*.

(v) A group of 6x6 matrices of size 720 in characteristic 2 isomorphic to S_6.

(vi) Take representative of orbit of length 35 and find its stabilizer in *grup*.

(vii) A group of 6x6 matrices of size 576 in characteristic 2 isomorphic to $2^4 : (S_3 \times S_3)$.

The Bibliography

- Faryad Ali *Fischer-Clifford Theory and Character Tables of Group Extensions*, PhD Thesis, University of Natal 2001
- J. H. Conway, et al, *Atlas of Finite Groups*, Oxford University Press, Oxford, 1985.
- I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, San Diego, 1976.
- J. Moori, *On certain groups associated with the smallest Fischer group*, J. London Maths Soc, Vol **2** (1981), 61 - 67.
- E.Z. Mpono, *Fischer-Clifford Theory and Character Tables of Group Extensions*, PhD Thesis, University of Natal, 1998.
- N.S. Whitley, *Fischer Matrices and Character Tables of Group Extensions*, MSc Thesis, University of Natal, 1994.