Maximal Subgroups of Finite Groups

Groups St Andrews 2009 in Bath

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In honour of John Cannon and Derek Holt

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Primitive permutation groups

Theorem

- Any perm group is a subdirect product of transitive groups (action on each orbit).
- Any transitive group is a subgroup of an iterated wreath product of primitive groups (action on blocks).

Q: What are the primitive groups of low degree?

What are the primitive groups of low degree?

- 1871, $d \leq 17$: Jordan. Some omissions in degrees 9, 12, 15, 16, 17.
- By 1910, $d \leq 20$: Bennett, Cole, Martin, Miller.
- By 1970, $d \leq 50$: Sims. Database in (eventually) GAP and MAGMA.
- 1980s. Classification of Finite Simple Groups.
- 1988. Dixon/Mortimer. Primitive groups with insoluble socles of degree < 1000.
- 1991. Short. Primitive soluble groups of degree < 256.
- By 2001: GAP has all $d < 256$, plus Dixon and Mortimer's (Theißen). MAGMA only has $d < 256$.

The primitive groups of degree less than 1000

My first task from John Cannon: Tidy up the classification, and make into a MAGMA database.

Let $G \leq S_d$ be a primitive group with soluble socle, then G is of affine type and

- $d = p^n$ for some prime p;
- $G \cong V{:}H$ where $V \cong \mathbb{F}_p^n$ and $H \leq \mathrm{GL}(n, p)$;
- H acts irreducibly on V.

Suffices to classify the irreducible subs of $\mathrm{GL}(n, p)$ for $p^n < 1000$, up to conjugacy in $\mathrm{GL}(n, p)$.

Classifying groups of affine type

- Construct a list containing all irreducible maximal subgroups of $\mathrm{GL}(n, p)$. (More on this later)
- Compute their irreducible maximal subgroups ... and recurse.
- Delete GL-conjugate subgroups (CMRD 04), say H and K:
  - Find a common Aschbacher class $\mathcal{C}$, and corresponding geometries for H and K.
  - Conjugate H, K into the stabiliser C in $\mathrm{GL}(d, q)$ of that geometry, and check conjugacy inside C.
  - If conjugate, done.
  - Else find another way of conjugating K into C. If tried all possible ways, they're not conjugate.
- Time for $\mathrm{GL}(6, 3)$: down from $\sim$ 2 months to 20 minutes.

THEOREM (CMRD/Unger '03) The primitive groups of degree less than 1000 are known.

. . . and beyond 1000?

O'NAN-SCOTT THEOREM (sort of)
A primitive G with insoluble socle of degree $< 60^6$ is almost simple, or of diagonal or product action type.

Diagonal, product action: use GAP/MAGMA.

THEOREM (CMRD '05) The primitive groups of degree less than 2500 are known.

THEOREM (Coutts/Quick/CMRD '09) The primitive groups of degree less than 4096 are known.

Simple groups and maximal subgroups

- If G is almost simple with socle S then a maximal subgroup $M \leq G$ is:
  - ordinary if $S \cap M$ is maximal in S;
  - a novelty if $S \cap M$ is nonmaximal in S;
  - a triviality if $S \leq M$.
- To classify the almost simple primitive groups of degree d, must determine the almost simple groups G with an ordinary or novelty maximal subgroup of index d.
- Use work of Cooperstein, Landazuri, Liebeck, Seitz, ... to determine candidate S.
- Then need to find maximals of all G with socle S.

The almost simple groups database

MAGMA database containing information about almost simple groups G - designed by Holt.

- Includes all almost simple G whose socle S is:
  - order less than 16000000; or
  - M_{24}, HS, J_3, McL, $\mathrm{Sz}(32)$ or $\mathrm{PSL}(6,2)$.
- Stores standard generators x, y for S: gens that are uniquely determined, up to automorphisms of S.
- For each $M <_{\max} G$, stores words in x and y for the generators of $S \cap M$ and much other information.
- If all composition factors of a perm group P are in the database, then the subgroups of P can be calculated (all: Cannon/Cox/Holt 96; maximal: Cannon/Holt 04).

Life beyond databases

A second task from John Cannon: Extend the almost simple groups database.

But infinite task adding groups one by one, and I'm lazy.

New idea: Construct families of maximal subgroups of almost simple groups in their natural representation.
Use constructive recognition to map maximals to the input group.

Maximal subgroups of alternating groups

Liebeck/Praeger/Saxl 87 determine maximals of A_d and S_d.

- Primitive maximals: need almost simple primitive groups of degree d, but containments known (Liebeck/Praeger/Saxl 90).

Corollary (CMRD) The maximal subgroups of A_d and S_d are known for $d < 2500$.

Constructive recognition algorithms for A_d by Bratus/Pak and Beals et al.

- So the maximals can can be constructed in any representation.

Some constructive theorems

THEOREM (Holt/CMRD '05)
The geometric maximal subgroups of $\mathrm{SL}(n,q)$, $\mathrm{Sp}(n,q)$, $\mathrm{SU}(n,q)$ can be constructed in the natural representation, up to conjugacy in $\mathrm{GL}(n,q)$, $\mathrm{Sp}(n,q)$ or $\mathrm{GU}(n,q)$, in $O(n^3 \log n \log^3 q)$ field operations.

THEOREM (Holt/CMRD '09)
Let G be s.t. $\mathrm{P}\Omega^\epsilon(d,q) \leq G \leq \mathrm{Aut}(\mathrm{P}\Omega^\epsilon(n,q))$, where $n \geq 7$. Then canonical generators of the pre-image in $\Omega^\epsilon(n,q)$ of the intersection with $\mathrm{P}\Omega^\epsilon(n,q)$ of the geometric maximal subgroups of G, up to conjugacy in $\mathrm{N}_{\mathrm{GL}(n,q)}(\Omega^\epsilon(n,q))$, can be constructed in $O(n^3 + n^2 \log q + \log q \log\log q)$.

- Holt has built MAGMA database of representations of almost simple groups in dimension up to 12 (+ many more).

Which of the candidate maximals **are** maximal?

- Kleidman/Liebeck determine maximality of geometric subgroups in dimension at least 13

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Which of the candidate maximals **are** maximal?

- Kleidman/Liebeck determine maximality of geometric subgroups in dimension at least 13, and give candidate geometric maximals in all dimensions where Aschbacher's theorem applies.
- Various old results for low dimensions, but often contain errors.
- Lübeck '01 and Hiss/Malle '01 give socles of potentially maximal $\mathcal{C}_9$ groups.

An ongoing project

Classify the maximal subgroups of the almost simple classical groups in dimension at most 12 (with Bray and Holt).

A sample issue:
LEMMA (Bray/Holt/CMRD 09) When $n > 2$ is even and p is odd, there are two isomorphism classes of groups $\mathrm{SU}(n, p^e) : \langle\phi\rangle$, where ϕ induces the natural Galois automorphism on matrix entries.

Executive summary:

- Geometric subgroups behave themselves, at least for $n \geq$ 6ish.
- $\mathcal{C}_9$ subgroups do not – many novelties.

London Math. Soc. Lecture Note Series, to appear.

The End

Happy birthday John and Derek!