Growth of the number of generators of direct products

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Setting

$d(G) =$ min. no. generators of G.

Problem

For fixed G, describe the behaviour of $d(G^n)$ as n increases.

$$d(C_m^n) = n \qquad \text{for all } n.$$

P.Hall (1936)

The Eulerian function of a group: establishes info that tells us

$$d(A_5) = 2, \quad \ldots, \quad d(A_5^{19}) = 2, \quad d(A_5^{20}) = 3, \quad \ldots$$

Wiegold & coauthors (1970s/1980s)

For G finite group

$$d(G^n) \text{ grows logarithmically} \iff G \text{ perfect}$$
$$d(G^n) \text{ grows linearly} \iff G \text{ not perfect}$$

For G infinite, situation more complicated!

(For G not perfect, still linear growth)

G f.g. infinite simple: $d(G) \leqslant d(G^n) \leqslant d(G) + 1$ always

A new viewpoint

Fact from a universal algebra setting

A non-abelian finite simple group G is functionally complete:

If $f\colon G^n \to G$ is any function, then f is equal to a "polynomial":

$$f(g_1, \dots, g_n) = w(g_1, \dots, g_n, c_1, \dots, c_m)$$

where w is a group word and the $c_i \in G$ are constants.

Proof of Wiegold's result for G finite non-abelian simple

Fix d. Put $n = |G|^d$. List all d-tuples:

$$\boldsymbol{y}_1 = (y_{11}, y_{21}, \dots, y_{d1})$$
$$\vdots$$
$$\boldsymbol{y}_n = (y_{1n}, y_{2n}, \dots, y_{dn})$$

Let

$$\boldsymbol{x}_i = (y_{i1}, y_{i2}, \dots, y_{in}) \in G^n \qquad \text{for } 1 \leqslant i \leqslant d.$$

Claim: $G^n = \langle \boldsymbol{x}_1, \dots, \boldsymbol{x}_d, \Delta(G) \rangle$
where $\Delta(G)$ is the diagonal copy of G.

Theorem

$d(G^{|G|^d}) \leqslant d + d(G)$. $\qquad$ $d(G^n)$ *grows logarithmically.*

Proof: $G^n = \langle \boldsymbol{x}_1, \dots, \boldsymbol{x}_d, \Delta(G) \rangle$

Let $(a_1, \dots, a_n) \in G^n$.
Define $f\colon G^d \to G$ by $\boldsymbol{y}_j \mapsto a_j$.
This is a polynomial function: there are a word w and constants $c_k \in G$ s.t.

$$w(y_{1j}, y_{2j}, \dots, y_{dj}, c_1, \dots, c_m) = a_j$$

Let $\hat{c}_k = (c_k, \dots, c_k) \in \Delta(G) \leqslant G^n$. Evaluate w in G^n:

$$\begin{aligned} w(\boldsymbol{x}_1, \dots, \boldsymbol{x}_d, \hat{c}_1, \dots, \hat{c}_m) &= (w(y_{11}, y_{21}, \dots, y_{d1}, c_1, \dots, c_m), \dots) \\ &= (a_1, \dots, a_n). \end{aligned}$$

So $(a_1, \dots, a_n) \in \langle \boldsymbol{x}_1, \dots, \boldsymbol{x}_d, \Delta(G) \rangle$. □

Consequence: a theorem about functionally complete algebraic structures!

Summary for simple finite classical algebraic structures:

	logarithmic (funct. complete)	linear (not funct. complete)
groups	non-abelian	abelian (C_p)
rings	with 1	no 1
modules	—	all
F-algebras	with 1	no 1
Lie algebras	non-abelian, $L = [L, L]$	abelian

Lifting to perfect groups

Theorem (Gaschütz, 1955)

If $\phi\colon G \to H$ is a surjective homomorphism and $d(G), d(H) \leqslant n$, then every generating set $\{y_1, \dots, y_n\}$ for H can be lifted to a generating set for G.

Also not a theorem about groups!
Holds in a congruence-uniform variety of algebraic structures, e.g., whenever homomorphic images correspond to quotients.

Summary

Growth of $d(G^n)$ for non-trivial finite algebraic structures:

	logarithmic	linear
groups	perfect	not perfect
rings	with 1	no 1
modules	—	all
F-algebras	with 1	no 1
Lie algebras	perfect, $L = [L, L]$	not perfect

Analogues of Wiegold's results on the infinite

Theorem

If A is an infinite, finitely generated simple group, ring, F-algebra or Lie algebra, then the sequence $(d(A^n))_{n \in \mathbb{N}}$ is eventually constant. Specifically,

$$d(A) \leqslant d(A^n) \leqslant d(A) + 1 \qquad \text{for } n \in \mathbb{N}.$$

- Uses *interpolation property* rather than *functional completeness*.
- $A^n = \langle \Delta(A), \boldsymbol{a} \rangle$ for any $\boldsymbol{a}$ with distinct entries.
- Finitely generated infinite simple algebraic structures are of significance throughout algebra. Such rings less common in the literature!

  Construct an infinite finitely generated simple ring without 1.
- $d(A^n) = 2$ if A is an Artinian simple algebra over an algebraically closed field (by Artin–Wedderburn, . . .).

Analogues of Wiegold's results on the infinite, II

Theorem

If A is one of

- *a finitely generated perfect group,*
- *a finitely generated ring with 1,*
- *a finitely generated F-algebra with 1, or*
- *a finitely generated perfect Lie algebra,*

then

$(d(A^n))_{n\in\mathbb{N}}$ *is bounded above by a logarithmic function of n.*

Open problems still remain throughout (including groups).

Thank you for your attention!