The Möbius Numbers of the Symmetric Groups: Using Orbits on Pairs

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Definitions and The Question

- A set L is a **lattice** iff it is a partially-ordered set where every pair of elements have a unique greatest lower bound called the meet and a unique least upper bound called the join.
- The **Möbius function** of a lattice L:

$$\sum_{z\in[x,y]} \mu_L(x,z) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

- The **Möbius number** of a lattice: Since a lattice has a unique max and min, the Möbius number of a lattice L with min 0 and max 1 is $\mu(L) = \mu_L(0,1)$.
- Example: The number-theoretic Möbius function $\mu(n)$ is just the Möbius number of the lattice of divisors of n.

The Groups and The Question

- **Möbius number of a group**: If G is a group and $\mathcal{L}(G)$ is the lattice of subgroups of G, $\mu(G) = \mu(\mathcal{L}(G))$.
- What is $\mu(S_n)$?

Some Small Values of n

n	2	3	4	5	6	7	8	9	10	11
$\mu(S_n)$	$\frac{-2!}{2}$	$\frac{3!}{2}$	$\frac{-4!}{2}$	$\frac{5!}{2}$	$-6!$	$\frac{7!}{2}$	$\frac{-8!}{2}$	$\frac{9!}{2}$	$\frac{-10!}{2}$	$\frac{11!}{2}$

One More Small Value of n

n	2	3	4	5	6	7	8	9	10	11	12
$\mu(S_n)$	$\frac{-2!}{2}$	$\frac{3!}{2}$	$\frac{-4!}{2}$	$\frac{5!}{2}$	$-6!$	$\frac{7!}{2}$	$\frac{-8!}{2}$	$\frac{9!}{2}$	$\frac{-10!}{2}$	$\frac{11!}{2}$	$-12!$

Infinite Families

The value of $\mu(S_n)$ has been computed for three infinite families, namely n prime, n twice a prime, or n a power of 2.

Theorem (Pahlings 1995 and Shareshian 1997)

Let n be a prime or a power of two. Then

$$\mu(S_n) = (-1)^{n-1} \frac{n!}{2}$$

Theorem (Shareshian 1997)

Let $n = 2p$ where p is a prime. Then

$$\mu(S_n) = \begin{cases} -n! & \textit{if } n-1 \textit{ is prime and } p \equiv 3 \pmod 4 \\ \frac{n!}{2} & \textit{if } n = 22 \\ -\frac{n!}{2} & \textit{otherwise} \end{cases}$$

Main Tool: A Closure Operation

- Suppose we have a map from L to L, written with an overline, satisfying the following three properties for all $x, y \in L$:

$$(i)\ x \leq \overline{x}$$
$$(ii)\ \overline{x} = \overline{\overline{x}}$$
$$(iii)\ x \leq y \Rightarrow \overline{x} \leq \overline{y}$$

  Such a map is called a ***closure operator***.
- If $x \in L$ has $\overline{x} = x$, we say that x is ***closed***.
- Given a closure operator, we can take the sublattice consisting of only the closed elements of L, which we call the ***quotient lattice*** $\overline{L}$.

Crapo's Closure Theorem

Theorem (Crapo 1969)

Let overline be a closure operator on L where $\overline{0} = 0$ is the min and 1 is the max. Then

$$\sum_{\overline{z}=1} \mu_L(0, z) = \mu\left(\overline{L}\right).$$

Closure on Orbits

- Define the closure operation overline on $\mathcal{L}(S_n)$ for any $G \leq S_n$ by $\overline{G} = S(\mathcal{O}_1) \times S(\mathcal{O}_2) \times ... \times S(\mathcal{O}_m)$ where $\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_m$ are the orbits of G and $S(\mathcal{O}_i)$ is the symmetric group on $\mathcal{O}_i$.
- $\overline{\langle(1,2)(3,4)\rangle} = \langle(1,2),(3,4)\rangle$
- Closed subgroups correspond to partitions of the set $\{1,2,\ldots,n\}$; thus the quotient lattice $\overline{\mathcal{L}(S_n)} \cong \prod_n$ the lattice of partitions.
- $\mu(\prod_n) = (-1)^{n-1}(n-1)!$

A Useful Sum

Applying Crapo's Closure Theorem:

$$\sum_{\substack{G \leq S_n \\ G \text{ transitive}}} \mu(G) = (-1)^{n-1}(n-1)!$$

A Different Closure Operation: Using Pairs

- Given any permutation representation of a group G on a set X, we get a new permutation representation on the set of ordered pairs of distinct elements from X by simply acting coordinate-wise.
- $[1, 2]^{(1,2,3)} = [2, 3]$
- Define a new closure operation written by an overline as follows: given $G \leq S_n$, $\overline{G}$ is the largest subgroup of S_n having the same orbits as G on ordered pairs of distinct points from $\{1, 2, \ldots, n\}$.
- $\overline{\langle (1, 2)(3, 4) \rangle} = \langle (1, 2)(3, 4) \rangle$
- $\overline{A_4} = S_4$ because A_4 can map any pair to any other pair.
- Closed subgroups are the 2-closed subgroups of S_n.

Another Useful Sum

Let $\mu_2(G)$ be the Mobius number of the lattice of 2-closed subgroups of G. Applying Crapo's Closure Theorem:

$$\sum_{\substack{G \leq S_n \\ G \text{ 2-transitive}}} \mu(G) = \mu_2(S_n)$$

Where This Gets Us

- The good: there are FAR fewer 2-transitive subgroups than transitive subgroups.
- The bad: we don't know $\mu_2(S_n)$.
- The ugly: the lattice of 2-closed subgroups is ugly.
- How to get around this: On the lattice of 2-closed subgroups, apply the original closure operation on orbits.

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Yet another sum...

- As before, define the closure operation overline on the lattice of 2-closed subgroups of S_n for any $G \leq S_n$ by $\overline{G} = S(\mathcal{O}_1) \times S(\mathcal{O}_2) \times ... \times S(\mathcal{O}_m)$ where $\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_m$ are the orbits of G and $S(\mathcal{O}_i)$ is the symmetric group on $\mathcal{O}_i$.
- $\overline{\langle (1,2)(3,4) \rangle} = \langle (1,2), (3,4) \rangle$
- One can ignore the second coordinate in an orbit on pairs, so the 2-closed subgroups still correspond to partitions of the set $\{1, 2, \ldots, n\}$; thus the quotient lattice is again the lattice of partitions.
- $\mu(\prod_n) = (-1)^{n-1}(n-1)!$

...but I promise it will be worth it.

Applying Crapo's Closure Theorem one last time:

$$\sum_{\substack{G \leq S_n \\ G \text{ transitive and 2-closed}}} \mu_2(G) = (-1)^{n-1}(n-1)!$$

μ and μ_2 Values for Some 2-Closed Transitive Subgroups

m	2	3	4	5
$\mu(S_2 \wr S_m)$	0	48	0	1920

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An Infinite Family of μ_2 Values

Theorem (M. 2009)

$\mu_2(S_2 \wr S_m) = 0$ *for* $m \geq 2$.

Proved using Crapo's Complement Theorem.

Crapo's Complement Theorem

If x, y are elements of a lattice L with $x \wedge y = \widehat{0}_L$ and $x \vee y = \widehat{1}_L$, we say that x is a ***complement*** of y, or x ***complements*** y.

Theorem (Crapo 1969)

Let L be a lattice. If there exists $x \in L$ such that x has no complement in L, then $\mu(L) = 0$.

Sketch of Proof of Theorem: The diagonal subdirect product of $G = (S_2)^m$ is a 2-closed subgroup of $S_2 \wr S_m$. Any complement of G in the full subgroup lattice would have to have the same orbits on pairs as $S_2 \wr S_m$.

Future Work

- Extend result on wreath products to $\mu_2(S_n \wr S_m)$ and other 2-closed transitive groups.
- Use that to compute $\mu_2(S_n)$ and in turn $\mu(S_n)$ by plugging into:

$$\sum_{\substack{G \leq S_n \\ G \text{ 2-transitive}}} \mu(G) = \mu_2(S_n)$$

- It is known that $\mu(S_n)$ is always divisible by $n!/2$ [Kratzer and Thévenaz 1984]. Figure out if and why $\mu_2(S_n)$ is a multiple of $\frac{n!}{n-1}$ for $n \geq 3$.

n	2	3	4	5	6	7	8	9
$\mu_2(S_n)$	-1	$\frac{3!}{2}$	$-\frac{4!}{3}$	0	0	$-\frac{7!}{6}$	$-2 \cdot \frac{8!}{7}$	0

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Gratitude

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