Engel’s law is an observation in economics stating that, with a given set of tastes, as income rises, the proportion of income spent on food falls.
On Engel and positive laws
What do they have in common?

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Notation
\( \mathfrak{A}_p \) – the variety of all abelian groups of exponent \( p \),
$\mathcal{A}_p$ – the variety of all abelian groups of exponent $p$,

$\mathcal{N}_c$ – the variety of all nilpotent groups of nilpotency class $c$, 

$F = \langle x, y \rangle$,

$[x, y] = x^{-1}y^{-1}xy$,

$x^iy = y^{-i}x$,

$[x, i + 1]y = [[x, i]y, y]$,

$E_n = \langle [x, i]y, 0 \leq i \leq n \rangle$,

$E = \langle [x, i]y, 0 \leq i \rangle$. 

$A_p$ – the variety of all abelian groups of exponent $p$,
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$S_d$ – the variety of all soluble groups of solubility length $d$. 
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(the variety generated by all finite groups of exponent \( e \)).
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By Zelmanov’s positive solution of the Restricted Burnside Problem all groups in $B_e$ are locally finite of exponent dividing $e$. 
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\begin{align*}
F &= \langle x, y \rangle, \quad [x, y] = x^{-1}y^{-1}xy, \quad x^{y^i} = y^{-i}x y^i.
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The Engel laws
Friedrich Engel (1861-1941, Germany) studied under Felix Klein at Leipzig, worked in Lie algebras.

The law \[
[x, y^n] \equiv 1
\]
is called the \(n\)-Engel law.

1936: M. Zorn: every finite Engel group is nilpotent. The \(n\)-Engel law does not imply nilpotency when \(n > 2\).

1971: S. Bachmuth and H. Y. Mochizuki: \(\exists\) a non-soluble locally finite 3-Engel group of exponent 5 (\(c \leq 2^n - 1\)).

1997: M. Vaughan-Lee: 4-Engel groups of exponent 5 are locally nilpotent. All known \(n\)-Engel groups are locally nilpotent.
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All known $n$-Engel groups are locally nilpotent.
Question:

Is every $n$-Engel group locally nilpotent? (Is every finitely generated $n$-Engel group nilpotent?)

There are two approaches:

1. $n$-Engel groups are locally nilpotent if:
   - 1942: $n = 2$ – F. W. Levi,
   - 1961: $n = 3$ – H. Heineken,
   - 2005: $n = 4$ – G. Havas and M. R. Vaughan-Lee,
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Does there exist a f.g. infinite simple $n$-Engel group?
Positive laws
A law \( u(x_1, ..., x_n) \equiv v(x_1, ..., x_n) \) is called **positive** if \( u, v \) are written **without inverses of variables**.
A law \( u(x_1, \ldots, x_n) \equiv v(x_1, \ldots, x_n) \) is called positive if \( u, v \) are written without inverses of variables.

We say that \( G \) is a \( p.l.-\)group if \( G \) satisfies a positive law.
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We say that \( G \) is a \( p.l.-\text{group} \) if \( G \) satisfies a positive law.

If \( G \) satisfies a law \( x^k \equiv 1 \) we call it a group of finite exponent.
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If $G$ satisfies a law $x^k \equiv 1$ we call it a group of finite exponent.

Each positive law implies a binary positive law $u(x, y) \equiv v(x, y)$ if substitute $x_i \rightarrow xy^i$. 
A law \( u(x_1, ..., x_n) \equiv v(x_1, ..., x_n) \) is called positive if \( u, v \) are written without inverses of variables.

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The law \( xy^2x \equiv yx^2y \) is cancelled, balanced and of degree 4.
A law $u(x_1, \ldots, x_n) \equiv v(x_1, \ldots, x_n)$ is called positive if $u$, $v$ are written without inverses of variables.

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Note that $p.l.$- groups do not contain free subsemigroups.
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The law $P_n \equiv Q_n$ defines $n$-nilpotent groups:
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The law $PzQ \equiv QzP$ implies:
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Then $PQ^{-1}z \equiv zPQ^{-1}$, $[PQ^{-1}, z] \equiv 1$. 

Corollary
Nilpotent-by-(finite exponent) groups satisfy positive laws.
The law $PzQ \equiv QzP$ implies: $Q^{-1}Pz \equiv zPQ^{-1}$, $PQ \equiv QP$.

Then $PQ^{-1}z \equiv zPQ^{-1}$, $[PQ^{-1}, z] \equiv 1$. So $PQ^{-1}$ is in $Z(G)$,
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Then $G$ is nilpotent of class $n$.

If $G / N$ satisfies $x^k \equiv 1$ and $N$ satisfies a p.l. $u(x, y) \equiv v(x, y)$ then $G$ satisfies the p.l.

$$u(x^k, y^k) \equiv v(x^k, y^k).$$
The law $PzQ \equiv QzP$ implies: $Q^{-1}Pz \equiv zPQ^{-1}$, $PQ \equiv QP$.

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**Corollary**

*Nilpotent-by-(finite exponent) groups satisfy positive laws.*
Nilpotent-by-(finite exponent) groups satisfy $p$. The question whether every $p$. $l$. group must be nilpotent-by-(finite exponent)? was open since 1953. The counterexample was constructed in 1996 by Olshanskii and Storozhev.
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Residually finite groups
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- No laws
- Positive laws
- Nilpotent-by-finite gps
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- Residually finite groups

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It follows that a f.g. residually finite group satisfying an Engel law or a positive law is nilpotent-by-finite.
Residually finite groups satisfying Engel or positive laws.
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Locally graded groups
1970, S. N. Černikov:

Definition

G is locally graded if every nontrivial finitely generated subgroup of G has a proper subgroup of finite index.

The class of locally graded groups avoids groups such as infinite Burnside groups or Ol'shanskii-Tarski monsters.

A NON-(locally graded) group must contain a f.g. infinite simple section.

A group which has no f.g. infinite simple sections is locally graded.
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Locally graded groups

This class contains all soluble groups, locally finite groups, residually finite groups. It is closed under taking subgroups and extensions. It is also closed under taking groups which are locally- or residually- in this class.
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1994 – Kim and Rhemtulla:
1994 – Kim and Rhemtulla: let $G$ be a locally graded group, then:

- If $G$ is finitely generated satisfying a positive law, then $G$ is polycyclic-by-finite.
- If $G$ is an $n$-Engel group, then $G$ is locally nilpotent.

It follows that a locally graded group satisfying an Engel or a positive law is locally soluble-by-finite.
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Locally graded groups with Engel or positive laws are locally (soluble-by-finite) (*)
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In 1997: R. Burns, Yu. Medvedev and O.M. considered so called Class $C$ consisting of locally-(residually-$SB$), groups. It was shown:

If $G \in C$ satisfies a positive law of degree $n$ then $G \in N_{cBｅ}$, where $c$, $e$ depend on $n$ only.

1998: If $G \in C$ satisfies $n$-Engel law then $G \in N_{cBｅ}$, where $c$, $e$ depend on $n$ only.

So by (*) we get Corollary: If $G$ is a locally graded group satisfying an $n$-Engel law or a positive law of degree $n$ then $G \in N_{cBｅ}$, where $c$, $e$ depend on $n$ only.
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**Corollary**

If $G$ is a **locally graded** group satisfying an **$n$-Engel law** or a **positive law** of degree $n$ then $G \in \mathcal{N}_c\mathcal{B}_e$, where $c, e$ depend on $n$ only.
Every locally graded group satisfying either Engel or positive law is nilpotent-by-locally finite of finite exponent.
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Question by R.G. Burns
Every \textit{locally graded} group satisfying either \textit{Engel} or \textit{positive} law is nilpotent-by-locally finite of finite exponent.

\textbf{Question by R.G. Burns}

\textit{What do the Engel laws and positive laws have in common that forces finitely generated locally graded groups satisfying them to be nilpotent-by-finite?}
The answer is:
The answer is:

The Engel laws and positive laws have the same so called Engel construction.
Engel Construction of laws
Definition

Let $u$ be a word and $S$ be a subset in $F$. 

For example, the laws $[x, y] \equiv x^p$ have construction $[x, y] \tilde{\in} \{x^p, p \in P\}$. They define varieties $A_p$. The laws with construction $[x, y] \tilde{\in} F^{\prime\prime}$ define varieties of groups with perfect commutator subgroups (i.e. $G^\prime = G^{\prime\prime}$).
Definition

Let $u$ be a word and $S$ be a subset in $F$.

We say that a binary law $w \equiv 1$ has construction $u \in S$.

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The laws with construction $[x, y] \in F'$ define varieties of groups with perfect commutator subgroups (i.e. $G' = G''$).
Definition

Let $u$ be a word and $S$ be a subset in $F$. We say that a binary law $w \equiv 1$ has construction $u \tilde{\in} S$ if it is equivalent to a law $u \equiv s$ for some word $s \in S$.

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Construction of the laws: $u \sim S$

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$$x^{k_0}[x, y]^{k_1}[x, 2y]^{k_2}...[x, ny]^{k_n}, \quad k_i \in \mathbb{Z}.$$
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Construction of the laws: \( u \in S \)

We speak of the Engel Construction if \( u \) is of the form

\[
x^{k_0} [x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, ny]^{k_n}, \quad k_i \in \mathbb{Z}.
\]

and \( S \) is a subset of \( E' \), where \( E = \langle [x, iy], \ 0 \leq i \rangle \).

We can show that every law has the General Engel Construction:

\[
x^{k_0} [x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, ny]^{k_n} \in \mathbb{E}'
\]
To show that every binary law has the General Engel Construction, we need a technical Lemma, which states:
$E_n = \langle [x, iy], 0 \leq i \leq n \rangle, \quad E = \langle [x, iy], 0 \leq i \rangle.$

To show that every binary law has the General Engel Construction, we need a technical Lemma, which states:

$$\langle [x, iy], 0 \leq i \leq n \rangle = \langle x, [x, y^i], 0 < i \leq n \rangle = \langle x^{y^i}, 0 \leq i \leq n \rangle.$$
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\[ [x, y^n] \in E_{n-1}[x, ny], \]
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So we have

\[ E_n = \langle x^y^i, \ 0 \leq i \leq n \rangle \quad \text{and} \quad E = \langle x^y^i, \ 0 \leq i \rangle. \]
\[ E = \langle [x, iy], 0 \leq i \rangle = \langle x^{y^i}, 0 \leq i \rangle. \]
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**Theorem**

*Every binary law* \( w \equiv 1 \) *has the General Engel Construction*

\[
x^{k_0} [x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, ny]^{k_n} \tilde{\in} E'.
\]
\[ E = \langle [x, iy], \ 0 \leq i \rangle = \langle x^y, \ 0 \leq i \rangle. \]

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\[ x^{k_0} [x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, ny]^{k_n} \in E'. \]

**Proof.** Let \( w \in F'. \) Since \( F' \subseteq \langle x \rangle^F, \) \( w \) is a product of some \( x^{y^i} \) with say, \( -m \leq i. \)
Every binary law $w \equiv 1$ has the General Engel Construction

$$x^{k_0}[x, y]^{k_1}[x, 2y]^{k_2}...[x, ny]^{k_n} \in \mathcal{E}'.$$

Proof. Let $w \in F'$. Since $F' \subseteq \langle x \rangle^F$, $w$ is a product of some $x^{y^i}$ with say, $-m \leq i$. Conjugation by $y^m$ gives us the equivalent law with $w \in \langle x^{y^i}, 0 \leq i \rangle = E$. 
\[ E = \langle [x, iy], \ 0 \leq i \rangle = \langle x^y, \ 0 \leq i \rangle. \]

**Theorem**

*Every binary law \( w \equiv 1 \) has the General Engel Construction*

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Theorem

Every binary law $w \equiv 1$ has the General Engel Construction

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$$[x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, ny]^{k_n} \tilde{\in} E'.$$
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**Theorem**

*Every binary law $w \equiv 1$ has the General Engel Construction*

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$$[x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, ny]^{k_n} \widetilde{\in} E'.$$

Now we add $x^{k_0}$ to get the required construction.
$K$-laws
\[x^{k_0}[x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, ny]^{k_n} \in \tilde{S} \subseteq E'.\]
We consider construction with \( k_n = 1 \) and \( S = E'_{n-1} \), that is
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x^{k_0}[x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, n-1y]^{k_{n-1}}[x, ny] \in E'_{n-1}.
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If \( n = 1 \) we have only one type of laws \( x^k[x, y] \equiv 1 \) defining varieties \( \mathcal{A}_k \).
A law is called an $R$-law if it implies a law with the Engel Construction:

$x^k_0 [x, y]^{k_1} [x, 2y]^{k_2} ... [x, (n-1)y]^{k_{n-1}} \tilde{\in} E'_{n-1}$,

or shortly $[x, ny] \tilde{\in} E_{n-1}$.

Clearly, $n$-Engel law is the $R$-law.

It can be shown that a positive law is the $R$-law.
A law is called an $\mathcal{R}$-law if it implies a law with the Engel Construction

$$x^{k_0}[x, y]^{k_1}[x, 2y]^{k_2} \cdots [x, n-1y]^{k_{n-1}}[x, ny] \in E'_{n-1}, \quad n \in \mathbb{N}, \quad k_i \in \mathbb{Z},$$
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\( \mathcal{R} \)-laws

**Definition**

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A law is called an $\mathcal{R}$-law if it implies a law with the Engel Construction

\[ x^{k_0}[x, y]^{k_1} [x, 2y]^{k_2} \cdots [x, n-1y]^{k_{n-1}} [x, ny] \sim E'_{n-1}, \quad n \in \mathbb{N}, \quad k_i \in \mathbb{Z}, \]

or shortly

\[ [x, ny] \sim E_{n-1}. \]

Clearly, $n$-Engel law is the $\mathcal{R}$-law.

It can be shown that a positive law is the $\mathcal{R}$-law.
Why "R"?

1968: J. Milnor considered f.g. groups with the property: for all \( g, h \in G \) the subgroup \( \langle g^h i, i \in \mathbb{Z} \rangle \) is f.g.

This property is called the Milnor property by F. Point.

In 1976 Rosset proved that each group without free non-cyclic subsemigroups has this property.

1994 - Kim and Rhemtulla call the groups with this property restrained.

So the groups satisfying positive laws are restrained.

We say that a law is restraining if it provides the above property.

We show that a law is restraining if and only if it is an \( R \)-law.
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So the groups satisfying positive laws are restrained.

We say that a law is restraining if it provides the above property.

We show that a law is restraining if and only if it is an \( \mathcal{R} \)-law.
\( R \) means restraining

Recall: An \( R \)-law is the law implying a law with the Engel Construction \([x, y], k \] \([x, 2y], k \) \([x, n-1y], k_{n-1} \[x, ny], \tilde{\sim} \in E_{n-1}, [x, n y], \tilde{\sim} \in E_n \).

**Theorem** A law \( w \equiv 1 \) is an \( R \)-law if and only if in every group \( G \) satisfying this law for all \( g, h \in G \) the subgroup \( \langle gh, i \rangle, i \in N \rangle \) is finitely generated.
Recall: An $R$-law is the law implying a law with the Engel Construction $[x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, n-1y]^{k_n-1} [x, ny] \in E'_{n-1}$, 

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Recall: An $\mathcal{R}$-law is the law implying a law with the Engel Construction $[x, y]^{k_1} [x, 2y]^{k_2} \ldots [x, n_1y]^{k_{n_1}} [x, ny] \in E_{n-1}$,

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**Theorem**

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\[[x, ny] \tilde{\in} E_{n-1}.\]

Proof.
Proof. If use $[x, y^n] \in E_{n-1}[x, ny]$ then we get $[x, y^n] \tilde{\in} E_{n-1}$, $x^{y^n} \tilde{\in} E_{n-1}$.
\([x, \, ny] \tilde{\in} \, E_{n-1}\).

**Proof.** If use \([x, \, y^n] \in E_{n-1}[x, \, ny]\) then we get \([x, \, y^n] \tilde{\in} \, E_{n-1}\), \(x^{y^n} \tilde{\in} \, E_{n-1}\). Since \(E_{n-1} = \langle x^y, \, 0 \leq i \leq n-1 \rangle\), we have
\[ [x, ny] \tilde{\in} E_{n-1}. \]

**Proof.** If use \([x, y^n] \in E_{n-1}[x, ny]\) then we get \([x, y^n] \tilde{\in} E_{n-1}, x^{y^n} \tilde{\in} E_{n-1}.\) Since \(E_{n-1} = \langle x^y, 0 \leq i \leq n-1 \rangle,\) we have \(x^{y^n} \tilde{\in} \langle x, x^y, x^{y^2}, \ldots, x^{y^{n-1}} \rangle.\)
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\[ x^{y^n} \tilde{\in} E_{n-1}. \]
Since \(E_{n-1} = \langle x^y, 0 \leq i \leq n-1 \rangle,\) we have
\[ x^{y^n} \tilde{\in} \langle x, x^y, x^{y^2}, ..., x^{y^{n-1}} \rangle. \]
Conjugation by \(y^{-n}\) gives \(x \tilde{\in} \langle x^{y^{-n}}, x^{y^{-(n-1)}}, ..., x^{y^{-2}}, x^{y^{-1}} \rangle,\)
Proof. If use \([x, y^n] \in E_{n-1}[x, ny]\) then we get \([x, y^n] \in E_{n-1}, xy^n \in E_{n-1}\). Since \(E_{n-1} = \langle x^y, 0 \leq i \leq n-1 \rangle\), we have
\[xy^n \in \langle x, xy, xy^2, ..., xy^{n-1} \rangle.\]
Conjugation by \(y^{-n}\) gives \(x \in \langle xy^{-n}, xy^{-(n-1)}, ..., xy^{-2}, xy^{-1} \rangle\), if change \(y \to y^{-1}\), then \(x \in \langle xy, xy^2, ..., xy^{(n-1)}, xy^n \rangle\).
\[ [x, n y] \tilde{\in} E_{n-1}. \]

**Proof.** If use \([x, y^n] \in E_{n-1}[x, n y] \) then we get \([x, y^n] \tilde{\in} E_{n-1}, x^{y^n} \tilde{\in} E_{n-1}. \) Since \( E_{n-1} = \langle x^{y^i}, 0 \leq i \leq n-1 \rangle, \) we have
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Conjugation by \( y^{-n} \) gives \( x \tilde{\in} \langle x^{y^{-n}}, x^{y-(n-1)}, \ldots, x^{y-2}, x^{y-1} \rangle, \)
\]
if change \( y \rightarrow y^{-1}, \) then \( x \tilde{\in} \langle x^y, x^{y^2}, \ldots, x^{y^{(n-1)}}, x^y \rangle. \) Let \( G \) be a relatively free group, freely generated by \( a, b, \ldots, \) satisfying an \( \mathcal{R} \)-law, then \( a \in \langle a^b, a^{b^2}, \ldots, a^{b^{(n-1)}}, a^b \rangle. \)
\[ [x, ny] \in E_{n-1}. \]

**Proof.** If use \([x, y^n] \in E_{n-1}[x, ny]\) then we get \([x, y^n] \tilde{\in} E_{n-1}, x^n \tilde{\in} E_{n-1}.\) Since \(E_{n-1} = \langle x^y, 0 \leq i \leq n-1 \rangle,\) we have \(x^n \tilde{\in} \langle x, x^y, x^{y^2}, \ldots, x^{y^{n-1}} \rangle.\)

Conjugation by \(y^{-n}\) gives \(x \tilde{\in} \langle x^{y^{-n}}, x^{y^{-(n-1)}}, \ldots, x^{-2}, x^{-1} \rangle,\)

if change \(y \rightarrow y^{-1},\) then \(x \tilde{\in} \langle x^y, x^{y^2}, \ldots, x^{y^{(n-1)}}, x^n \rangle.\) Let \(G\) be a relatively free group, freely generated by \(a, b, \ldots,\) satisfying an \(\mathcal{R}\)-law, then \(a \in \langle a^b, a^{b^2}, \ldots, a^{b^{(n-1)}}, a^b \rangle.\) Conjugation by \(b^{-1}\) gives \(a^{b^{-1}} \in \langle a, a^b, \ldots, a^{b^{(n-2)}}, a^{b^{(n-1)}} \rangle \subseteq \langle a^b, a^{b^2}, \ldots, a^{b^{(n-1)}}, a^b \rangle.\)
\[ [x, ny] \tilde{\in} E_{n-1}. \]

**Proof.** If use \([x, y^n] \in E_{n-1}[x, ny]\) then we get \([x, y^n] \tilde{\in} E_{n-1},\) \(x^y^n \tilde{\in} E_{n-1}.\) Since \(E_{n-1} = \langle x^y^i, 0 \leq i \leq n-1 \rangle,\) we have \(x^y^n \tilde{\in} \langle x, x^y, x^y^2, ..., x^y^{n-1} \rangle.\)

Conjugation by \(y^{-n}\) gives \(x \tilde{\in} \langle x^y^{-n}, x^y^{-(n-1)}, ..., x^y^{-2}, x^y^{-1} \rangle,\)

if change \(y \rightarrow y^{-1},\) then \(x \tilde{\in} \langle x^y, x^y^2, ..., x^y^{(n-1)}, x^y^n \rangle.\) Let \(G\) be a relatively free group, freely generated by \(a, b, ...,\) satisfying an \(R\)-law, then \(a \in \langle a^b, a^b^2, ..., a^b^{(n-1)}, a^b^n \rangle.\) Conjugation by \(b^{-1}\) gives \(a^{b^{-1}} \in \langle a, a^b, ..., a^b^{(n-2)}, a^b^{(n-1)} \rangle \subseteq \langle a^b, a^b^2, ..., a^b^{(n-1)}, a^b^n \rangle.\)

By repeating the conjugation by \(b^{-1}\) we obtain for all \(i \geq 0,\)
\[ a^{b^{-i}} \in \langle a^b, a^b^2, ..., a^b^{(n-1)}, a^b^n \rangle. \]
$$[x, ny] \in E_{n-1}.$$ 

**Proof.** If use $$[x, y^n] \in E_{n-1}[x, ny]$$ then we get $$[x, y^n] \tilde{\in} E_{n-1},$$ $$x y^n \tilde{\in} E_{n-1}.$$ Since $$E_{n-1} = \langle x^y, 0 \leq i \leq n-1 \rangle,$$ we have $$x y^n \tilde{\in} \langle x, x y, x y^2, ..., x y^{n-1} \rangle.$$ 

Conjugation by $$y^{-n}$$ gives $$x \tilde{\in} \langle x y^{-n}, x y^{-(n-1)}, ..., x y^{-2}, x y^{-1} \rangle,$$

if change $$y \rightarrow y^{-1},$$ then $$x \tilde{\in} \langle x y, x y^2, ..., x y^{(n-1)}, x y^n \rangle.$$

Let $$G$$ be a relatively free group, freely generated by $$a, b, ...,$$ satisfying an $$R$$-law, then $$a \in \langle a^b, a^{b^2}, ..., a^{b^{(n-1)}}, a^{b^n} \rangle.$$ Conjugation by $$b^{-1}$$ gives $$a^{b^{-1}} \in \langle a, a^b, ..., a^{b^{(n-2)}}, a^{b^{(n-1)}} \rangle \subseteq \langle a^b, a^{b^2}, ..., a^{b^{(n-1)}}, a^{b^n} \rangle.$$ 

By repeating the conjugation by $$b^{-1}$$ we obtain for all $$i \geq 0,$$ $$a^{b^{-i}} \in \langle a^b, a^{b^2}, ..., a^{b^{(n-1)}}, a^{b^n} \rangle.$$ 

$$\langle a^{b^i}, i \in \mathbb{Z} \rangle = \langle a^{b^{n-1}}, a^{b^{-(n-1)}}, ..., a^{b^{-1}}, a, a^b, ..., a^{b^{n-1}}, a^{b^n} \rangle$$ is f.g.
\([x, ny] \in E_{n-1}\).

**Proof.** If use \([x, y^n] \in E_{n-1}[x, ny]\) then we get \([x, y^n] \in E_{n-1}\), \(x^n \in E_{n-1}\). Since \(E_{n-1} = \langle x^y, 0 \leq i \leq n-1 \rangle\), we have
\[x^n \in \langle x, x^y, x^{y^2}, \ldots, x^{y^{n-1}} \rangle.\]
Conjugation by \(y^{-n}\) gives \(x \in \langle x^{y^n}, x^{y^{n-1}}, \ldots, x^{y}, x^{-1} \rangle\),
if change \(y \rightarrow y^{-1}\), then \(x \in \langle x^y, x^{y^2}, \ldots, x^{y^{n}}, x^y \rangle\). Let \(G\) be a relatively free group, freely generated by \(a, b, \ldots\), satisfying an \(R\)-law, then \(a \in \langle a^b, a^{b^2}, \ldots, a^{b^{(n-1)}}, a^b \rangle\). Conjugation by \(b^{-1}\) gives
\(a^{b^{-1}} \in \langle a^b, a^b, \ldots, a^{b^{(n-2)}}, a^{b^{(n-1)}} \rangle \subseteq \langle a^b, a^{b^2}, \ldots, a^{b^{(n-1)}}, a^b \rangle\).
By repeating the conjugation by \(b^{-1}\) we obtain for all \(i \geq 0\),
\(a^{b^{-i}} \in \langle a^b, a^{b^2}, \ldots, a^{b^{(n-1)}}, a^b \rangle\).
\(<a^b^i, i \in \mathbb{Z}> = \langle a^{b-n}, a^{b-(n-1)}, \ldots, a^{b-1}, a, a^b, \ldots, a^{b^{n-1}}, a^b \rangle\) is f.g.
Now use the fact, that \(a, b\) are the free generators.
What do the Engel laws and positive laws have in common?
Engel laws and positive laws are the $K$-laws

1. If $G$ is a finitely generated group and for all $g, h \in G$ the subgroup $\langle gh^i \rangle$, $i \in \mathbb{N}$, is f.g. then $G'$ is finitely generated.

2. If $G/N$ is cyclic then $N$ is finitely generated.

It follows:
Engel laws and positive laws are the $\mathcal{N}$-laws

1976, S. Rosset: if $G$ is a finitely generated group and for all $g, h \in G$ the subgroup $\langle g^h, i \in \mathbb{N} \rangle$ is f.g.
Engel laws and positive laws are the $R$-laws

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(i) $G'$ is finitely generated.
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It follows:
(ii) If $G/N$ is cyclic then $N$ is finitely generated.
(ii) If $G/N$ is cyclic then $N$ is finitely generated

Rosset Lemma

If $G$ is a finitely generated group satisfying an $R$-law, then:
(ii) If $G/N$ is cyclic then $N$ is finitely generated

Rosset Lemma

If $G$ is a finitely generated group satisfying an $\mathcal{K}$-law, then:

1. $G'$ is finitely generated,
(ii) If $G/N$ is cyclic then $N$ is finitely generated

Rosset Lemma

If $G$ is a finitely generated group satisfying an $R$-law, then:

1. $G'$ is finitely generated,

2. if $G/N$ is polycyclic then $N$ is finitely generated.
If $G/N$ is cyclic then $N$ is finitely generated.

Rosset Lemma

If $G$ is a finitely generated group satisfying an $R$-law, then:

1. $G'$ is finitely generated,

2. if $G/N$ is polycyclic then $N$ is finitely generated.

Proof

There is a finite subnormal series from $G$ to $N$ with cyclic factors.
(ii) If $G/N$ is cyclic then $N$ is finitely generated

Rosset Lemma

If $G$ is a finitely generated group satisfying an $\mathcal{R}$-law, then:

1. $G'$ is finitely generated,

2. if $G/N$ is polycyclic then $N$ is finitely generated.

Proof

There is a finite subnormal series from $G$ to $N$ with cyclic factors and by repeated application of result (ii) we obtain that $N$ is finitely generated.
Engel laws and positive laws are the $R$-laws

Theorem
A law is an $R$-law if and only if every f.g. group $G$ satisfying this law has its commutator subgroup $G'$ finitely generated.

Corollary
Every f.g. metabelian group $G$ satisfying an $R$-law is nilpotent-by-finite because by J. Groves, $G$ is either nilpotent-by-finite or $var G$ contains a subvariety $A_p A$ which contains $W = C_p wr C$ with $W'$ infinitely generated.
Engel laws and positive laws are the $R$-laws

**Theorem**

A law is an $R$-law if and only if every f.g. group $G$ satisfying this law has its commutator subgroup $G'$ finitely generated.
Engel laws and positive laws are the $\mathcal{R}$-laws

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Every f.g. metabelian group $G$ satisfying an $\mathcal{R}$-law is nilpotent-by-finite.
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A law is an $K$-law if and only if every f.g. group $G$ satisfying this law has its commutator subgroup $G'$ finitely generated.

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Every f.g. metabelian group $G$ satisfying an $K$-law is nilpotent-by-finite

because by J. Groves, $G$ is either nilpotent-by-finite or $\text{var } G$ contains a subvariety $\mathcal{A}_p\mathcal{A}$
Engel laws and positive laws are the $\mathcal{R}$-laws

**Theorem**

A law is an $\mathcal{R}$-law if and only if every f.g. group $G$ satisfying this law has its commutator subgroup $G'$ finitely generated.

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Every f.g. metabelian group $G$ satisfying an $\mathcal{R}$-law is nilpotent-by-finite

because by J. Groves, $G$ is either nilpotent-by-finite or $\text{var } G$ contains a subvariety $\mathcal{A}_p\mathcal{A}$ which contains $W = C_p \text{wr} C$ with $W'$ infinitely generated.
Engel laws and positive laws are the $K$-laws
Engel laws and positive laws are the $\mathcal{R}$-laws

Lemma (cf. 2003, Burns and Medvedev)

If every f.g. metabelian group satisfying a law $w \equiv 1$ is nilpotent-by-finite then:
every f.g. residually finite group satisfying the law $w \equiv 1$ is nilpotent-by-finite.
Moreover, the parameters $c, e$ depend on the law only.
Engel laws and positive laws are the $\mathcal{R}$-laws

Lemma (cf. 2003, Burns and Medvedev)

If every f.g. metabelian group satisfying a law $w \equiv 1$ is nilpotent-by-finite then:
every f.g. residually finite group satisfying the law $w \equiv 1$ is nilpotent-by-finite.
Moreover, the parameters $c, e$ depend on the law only.

Corollary

Every f.g. residually finite group $G$ satisfying an $\mathcal{R}$-law is nilpotent-by-finite.
Engel laws and positive laws are the $\mathcal{R}$-laws

We need one more property of $\mathcal{R}$-laws
Engel laws and positive laws are the $R$-laws

We need one more property of $R$-laws

**Lemma**

> In every f.g. group $G$ satisfying an $R$-law
> the finite residual $R$ is finitely generated.
Engel laws and positive laws are the R-laws

We need one more property of R-laws

**Lemma**

*In every f.g. group G satisfying an R-law the finite residual R is finitely generated.*

We use the fact that $G/R$ is nilpotent-by-finite,
Engel laws and positive laws are the $\mathcal{R}$-laws

We need one more property of $\mathcal{R}$-laws

**Lemma**

*In every f.g. group $G$ satisfying an $\mathcal{R}$-law, the finite residual $R$ is finitely generated.*

We use the fact that $G/R$ is nilpotent-by-finite, so it has a nilpotent subgroup $H/R$ which is f.g., hence polycyclic.
Engel laws and positive laws are the $\mathcal{R}$-laws

We need one more property of $\mathcal{R}$-laws

**Lemma**

*In every f.g. group $G$ satisfying an $\mathcal{R}$-law, the finite residual $R$ is finitely generated.*

We use the fact that $G/R$ is nilpotent-by-finite, so it has a nilpotent subgroup $H/R$ which is f.g., hence polycyclic. It follows by Rosset Lemma, that $R$ is finitely generated.
Now we can answer the question: What do the Engel laws and positive laws have in common that forces f.g. locally graded groups satisfying them to be nilpotent-by-finite?

The answer is: Engel laws and positive laws are the \( \mathbb{R} \)-laws and every \( \mathbb{R} \)-law forces f.g. locally graded groups satisfying it to be nilpotent-by-finite.

We show that every f.g. locally graded group satisfying an \( \mathbb{R} \)-law is nilpotent-by-finite.
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Now we can answer the question:
What do the Engel laws and positive laws have in common that forces f.g. locally graded groups satisfying them to be nilpotent-by-finite?

The answer is: Engel laws and positive laws are the $\mathcal{R}$-laws and every $\mathcal{R}$-law forces f.g. locally graded groups satisfying it to be nilpotent-by-finite.

We show that
Every f.g. locally graded group satisfying an $\mathcal{R}$-law is nilpotent-by-finite.
Every f.g. residually finite group satisfying an $\mathfrak{A}$-law is nilpotent-by-finite.
Every f.g. residually finite group satisfying an $\mathbb{R}$-law is nilpotent-by-finite.

**Theorem**

*Every f.g. locally graded group satisfying an $\mathbb{R}$-law is nilpotent-by-finite.*

**Proof.**
Every f.g. residually finite group satisfying an R-law is nilpotent-by-finite.

Theorem

Every f.g. locally graded group satisfying an R-law is nilpotent-by-finite.

Proof. $G$ is locally graded, $R$ is finitely generated.
Every f.g. residually finite group satisfying an $\mathcal{R}$-law is nilpotent-by-finite

**Theorem**

Every f.g. locally graded group satisfying an $\mathcal{R}$-law is nilpotent-by-finite.

**Proof.** $G$ is locally graded, $R$ is finitely generated. If $R \neq 1$, it must contain a proper subgroup of finite index $T \subset R$, say.
Every f.g. residually finite group satisfying an $\mathbb{R}$-law is nilpotent-by-finite

**Theorem**

Every f.g. locally graded group satisfying an $\mathbb{R}$-law is nilpotent-by-finite.

**Proof.** $G$ is locally graded, $R$ is finitely generated. If $R \neq 1$, it must contain a proper subgroup of finite index $T \varsubsetneq R$, say. We show that there exists $K \vartriangleleft G$, such that $K \subseteq T \varsubsetneq R$ and $|R : K| < \infty$. 
Every f.g. residually finite group satisfying an $\mathcal{R}$-law is nilpotent-by-finite

Theorem

Every f.g. locally graded group satisfying an $\mathcal{R}$-law is nilpotent-by-finite.

Proof. $G$ is locally graded, $R$ is finitely generated. If $R \neq 1$, it must contain a proper subgroup of finite index $T \lneq R$, say. We show that there exists $K \lhd G$, such that $K \subseteq T \lneq R$ and $|R : K| < \infty$.

Since $(G/K)/(R/K) \cong G/R$, 

Every f.g. residually finite group satisfying an $\mathfrak{A}$-law is nilpotent-by-finite.

**Theorem**

Every f.g. locally graded group satisfying an $\mathfrak{A}$-law is nilpotent-by-finite.

**Proof.** $G$ is locally graded, $R$ is finitely generated. If $R \neq 1$, it must contain a proper subgroup of finite index $T \subsetneq R$, say. We show that there exists $K \triangleleft G$, such that $K \subseteq T \subsetneq R$ and $|R : K| < \infty$.

Since $(G/K)/(R/K) \cong G/R$, $G/K$ is finite-by-(nilpotent-by-finite), hence $G/K$ is nilpotent-by-finite and then residually finite.
Every f.g. residually finite group satisfying an $R$-law is nilpotent-by-finite

**Theorem**

Every f.g. locally graded group satisfying an $R$-law is nilpotent-by-finite.

**Proof.** $G$ is locally graded, $R$ is finitely generated. If $R \neq 1$, it must contain a proper subgroup of finite index $T \subsetneq R$, say. We show that there exists $K \triangleleft G$, such that $K \subseteq T \subsetneq R$ and $|R : K| < \infty$.

Since $(G/K)/(R/K) \cong G/R$, $G/K$ is finite-by-(nilpotent-by-finite), hence $G/K$ is nilpotent-by-finite and then residually finite.

Then $R \subseteq K$, which contradicts to $K \subseteq T \subsetneq R$. 
Engel laws and positive laws are the $R$-laws.

Corollary:
For every $R$-law there exist positive integers $c$ and $e$ depending only on the law, such that every locally graded group satisfying this law lies in the product variety $N^c B^e$.

There are groups satisfying $R$-laws, which are not in any of $N^c B^e$:
- Burnside groups $B(r, n)$ for sufficiently large $n$,
- the groups satisfying the $R$-law $xy^n = y^n x$ also for $n$ sufficiently large.

Ol'shanskii and Storozhev groups which are not even locally soluble by-(finite exponent).

Problem: Is there an $R$-law that implies neither positive nor Engel law?
Engel laws and positive laws are the $\mathcal{R}$-laws

**Corollary**

For every $\mathcal{R}$-law there exist positive integers $c$ and $e$ depending only on the law, such that every locally graded group satisfying this law lies in the product variety $\mathcal{N}_c \mathcal{B}_e$. 
Engel laws and positive laws are the $R$-laws

**Corollary**

For every $R$-law there exist positive integers $c$ and $e$ depending only on the law, such that every locally graded group satisfying this law lies in the product variety $N_c B_e$.

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Engel laws and positive laws are the $R$-laws

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Engel laws and positive laws are the $\mathfrak{R}$-laws

**Corollary**

For every $\mathfrak{R}$-law there exist positive integers $c$ and $e$ depending only on the law, such that every locally graded group satisfying this law lies in the product variety $\mathfrak{N}_c \mathfrak{B}_e$.

There are groups satisfying $\mathfrak{R}$-laws, which are not in any of $\mathfrak{N}_c \mathfrak{B}_e$:

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**Corollary**

For every $R$-law there exist positive integers $c$ and $e$ depending only on the law, such that every locally graded group satisfying this law lies in the product variety $N_c B_e$.

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**Problem** Is there an $R$-law that implies neither positive nor Engel law?
Special kind of $\mathcal{R}$-laws
The construction \([x, ny] \in E_{n-1}\) defines the \(R\)-laws.

We consider the laws called \(L_n\) of the form \([x, ny] \equiv [x, n y], n > 1\).

**Proposition (i)**
Every metabelian group \(G\) satisfying \(L_n\) is abelian.

**Proposition (ii)**
Every finite group \(G\) satisfying \(L_n\) is abelian.

**Proof (i)**
If substitute \([y, n y - 1 x]\) for \(y\), we get \([x, [y, n y - 1 x]]]\) \(\equiv [x, n[y, n y - 1 x]] \in F''\).
By taking inverse and interchanging \(x \leftrightarrow y\) we obtain \([x, n y] \in F''\).
Now by \(L_n\) we have \([x, y] \in F''\). So \(G' = G'' = \{e\}\).

**Proof (ii)**
If there exist a non-abelian finite group satisfying \(L_n\).
Take such a group \(G\) of the smallest order.
By Miller and Moreno result (1903), a finite group \(G\), all whose proper subgroups are abelian, is metabelian.
Hence \(G\) must be abelian, a contradiction.
The construction \([x, ny] \in E_{n-1}\) defines the \(\mathcal{R}\)-laws.

We consider the laws called \(L_n\) of the form

\([x, y] \equiv [x, ny], \ n > 1\).
The construction \([x, \; n y] \in E_{n-1}\) defines the \(\mathcal{R}\)-laws.

We consider the laws called \(L_n\) of the form

\([x, y] \equiv [x, \; n y], \; n > 1\).

**Proposition**

(i) Every metabelian group \(G\) satisfying \(L_n\) is abelian.

(ii) Every finite group \(G\) satisfying \(L_n\) is abelian.
The construction \([x, ny] \tilde{\in} E_{n-1}\) defines the \(R\)-laws.

We consider the laws called \(L_n\) of the form

\([x, y] \equiv [x, ny], n > 1.\)

**Proposition**

(i) Every metabelian group \(G\) satisfying \(L_n\) is abelian.
(ii) Every finite group \(G\) satisfying \(L_n\) is abelian.

**Proof** (i) If substitute \([y, n-1x]\) for \(y\), we get
\([x, [y, n-1x]] \equiv [x, n[y, n-1x]] \in F''\).
The construction \([x, ny] \in E_{n-1}\) defines the \(R\)-laws.

We consider the laws called \(L_n\) of the form

\[
[x, y] \equiv [x, ny], \ n > 1.
\]

**Proposition**

(i) Every metabelian group \(G\) satisfying \(L_n\) is abelian.

(ii) Every finite group \(G\) satisfying \(L_n\) is abelian.

**Proof** (i) If substitute \([y, n_{-1}x]\) for \(y\), we get

\[
[x, [y, n_{-1}x]] \equiv [x, n[y, n_{-1}x]] \in F''.
\]

By taking inverse and interchanging \(x \leftrightarrow y\) we obtain \([x, ny] \in F''\).
The construction \([x, ny] \lessgtr E_{n-1}\) defines the \(R\)-laws.

We consider the laws called \(L_n\) of the form

\[[x, y] \equiv [x, ny], \ n > 1.\]

**Proposition**

(i) Every metabelian group \(G\) satisfying \(L_n\) is abelian.

(ii) Every finite group \(G\) satisfying \(L_n\) is abelian.

**Proof** (i) If substitute \([y, n_{-1}x]\) for \(y\), we get
\([x, [y, n_{-1}x]] \equiv [x, n[y, n_{-1}x]] \in F''\).

By taking inverse and interchanging \(x \lessgtr y\) we obtain \([x, ny] \lessgtr F''\).

Now by \(L_n\) we have \([x, y] \lessgtr F''\). So \(G' = G'' = \{e\}\).
The construction \([x, ny] \in E_{n-1}\) defines the \(\mathcal{R}\)-laws.

We consider the laws called \(L_n\) of the form

\([x, y] \equiv [x, ny], n > 1\).

**Proposition**

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**Proof** (i) If substitute \([y, n-1x]\) for \(y\), we get

\([x, [y, n-1x]] \equiv [x, n[y, n-1x]] \in F''\).

By taking inverse and interchanging \(x \leftrightarrow y\) we obtain \([x, ny] \in F''\).

Now by \(L_n\) we have \([x, y] \in F''\). So \(G' = G'' = \{e\}\).

(ii) If there exist a non-abelian finite group satisfying \(L_n\).
The construction $[x, ny] \# E_{n-1}$ defines the $R$-laws.

We consider the laws called $L_n$ of the form

$$[x, y] \equiv [x, ny], \ n > 1.$$

**Proposition**

(i) Every metabelian group $G$ satisfying $L_n$ is abelian.

(ii) Every finite group $G$ satisfying $L_n$ is abelian.

**Proof**

(i) If substitute $[y, n_{-1}x]$ for $y$, we get

$$[x, [y, n_{-1}x]] \equiv [x, n[y, n_{-1}x]] \in F''.$$

By taking inverse and interchanging $x \Leftrightarrow y$ we obtain $[x, ny] \# F''$. Now by $L_n$ we have $[x, y] \# F''$. So $G' = G'' = \{e\}$.

(ii) If there exist a non-abelian finite group satisfying $L_n$. Take such a group $G$ of the smallest order.
The construction \([x, ny] \bar{\in} E_{n-1}\) defines the \(\mathcal{R}\)-laws.

We consider the laws called \(L_n\) of the form

\[[x, y] \equiv [x, ny], \ n > 1.\]

**Proposition**

(i) Every metabelian group \(G\) satisfying \(L_n\) is abelian.

(ii) Every finite group \(G\) satisfying \(L_n\) is abelian.

**Proof**

(i) If substitute \([y, n_{-1}x]\) for \(y\), we get

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By taking inverse and interchanging \(x \Leftrightarrow y\) we obtain \([x, ny] \bar{\in} F''\).

Now by \(L_n\) we have \([x, y] \bar{\in} F''\). So \(G' = G'' = \{e\}\).

(ii) If there exist a non-abelian finite group satisfying \(L_n\). Take such a group \(G\) of the smallest order. By Miller and Moreno result (1903), a finite group \(G\), all whose proper subgroups are abelian, is metabelian.
The construction \([x, \; n y] \in E_{n-1}\) defines the \(\mathcal{R}\)-laws.

We consider the laws called \(L_n\) of the form

\[ [x, y] \equiv [x, \; n y], \; n > 1. \]

**Proposition**

(i) Every metabelian group \(G\) satisfying \(L_n\) is abelian.
(ii) Every finite group \(G\) satisfying \(L_n\) is abelian.

**Proof**

(i) If substitute \([y, \; n-1 x]\) for \(y\), we get

\[ [x, [y, \; n-1 x]] \equiv [x, \; n[y, \; n-1 x]] \in F'' . \]

By taking inverse and interchanging \(x \rightleftharpoons y\) we obtain \([x, \; n y] \in F''\).

Now by \(L_n\) we have \([x, y] \in F''\). So \(G' = G'' = \{e\}\).

(ii) If there exist a non-abelian finite group satisfying \(L_n\). Take such a group \(G\) of the smallest order. By Miller and Moreno result (1903), a finite group \(G\), all whose proper subgroups are abelian, is metabelian. Hence \(G\) must be abelian, a contradiction.
So every law $L_n$ of the form $[x, y] \equiv [x, ny]$, $n > 1$ is either abelian or pseudo-abelian.
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**Observation** *Let \(G\) satisfies the law \(L_n\). If an element \(b\) is conjugate to its inverse, \(b^{-1} = b^a\) say, then \(b^2 = 1\).*

**Proof** \[b^2 = (b^a)^{-1}b = [a, b] \equiv [a, nb] = [b^2, n^{-1}b] = 1.\]
For $n = 2$ and $n = 3$ we used the commutator identities (the equality in $F$) saying that \([x, ny^{-1}]\) is conjugate to \([x, ny]^{(-1)^n}\). However \([x, 4y^{-1}]\) is NOT conjugate to \([x, 4y]^{\pm 1}\). We conjecture that for $n > 3$ the law $[x, y] \equiv [x, ny]$ need not be abelian.
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