Engel conditions on orderable groups and in combinatorial problems

Marcel HERZOG, Patrizia LONGOBARDI, Mercede MAJ

TEL AVIV UNIVERSITY
UNIVERSITÀ DEGLI STUDI DI SALERNO

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Engel elements

$G$ a group, $x, y \in G$, $n$ a non-negative integer. The commutator $[x, ny]$ is defined, by induction, by the rules:

$$[x, 0y] = x, \quad [x, n+1y] = [[x, ny], y].$$

$x \in G$ is a right Engel element of $G$ (a left Engel element of $G$) if for each $g \in G$ there is an integer $n = n(x, g) \geq 0$ such that

$$[x, ng] = 1 \quad ([g, nx] = 1).$$

If $n$ can be chosen independently on $g$ we say that $x$ is a right $n$-Engel element (a left $n$-Engel element).

If every element of a group $G$ is a right Engel element (equivalently every element of $G$ is a left Engel element), then $G$ is called an Engel group. $G$ is called an $n$-Engel group if

$$[x, ny] = 1, \forall x, y \in G$$
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Remark

- \( G \) a nilpotent group of class \( c \) \( \implies \) \( G \) a \( c \)-Engel group.

- There exists an infinite 3-Engel group with trivial center, thus \( k \)-Engel groups need not to be nilpotent.

Remark

- \( G \) a finite \( k \)-Engel group \( \implies \) \( G \) nilpotent [M. Zorn, 1937]
- \( G \) a soluble \( k \)-Engel group \( \implies \) \( G \) locally nilpotent [K. Gruenberg, 1959]
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Question

Is every \( k \)-Engel group locally nilpotent?
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- **G a nilpotent group of class c** $\implies$ **G a c-Engel group.**
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Orderable Engel groups

$G$ a group, $\leq$ a partial order on the set $G$

$(G, \leq)$ is a **partially ordered group** if, for any $x, y, a, b \in G$,

$$x \leq y \implies axb \leq ayb.$$ 

If $(G, \leq)$ is a partially ordered group and the order $\leq$ is a total order in $G$, we say that $(G, \leq)$ is a **totally ordered group** or simply an **ordered group**.

$G$ is an **orderable group** (an **O-group**) if there exists a total order $\leq$ such that $(G, \leq)$ is an ordered group.

**Example**

Any nilpotent torsion-free group is an orderable group.
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\textbf{Example}

Any nilpotent torsion-free group is an orderable group.
Theorem (A)

[Y. K. Kim, A. H. Rhemtulla, 1995] An orderable k-Engel group is nilpotent of class \( \leq f(k) \).

It is very easy to see that an orderable group is always torsion-free and, as noticed before, every nilpotent torsion-free group is an orderable group, thus we could ask:

Question (A. I. Kokorin, problem 2.24 of The Kourovka Notebook)

Is every torsion-free k-Engel group an orderable group?

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Lemma (1)

Let $G$ be a $k$-Engel group, then the subgroup $\langle x \rangle \langle y \rangle$ can be generated by $k$ elements, for any $x, y \in G$.

Lemma (2)

Let $G$ be a finitely generated $k$-Engel group. If $H$ is normal in $G$ and $G/H$ is cyclic then $H$ is finitely generated.
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Let $G$ be a finitely generated $k$-Engel group. If $H$ is normal in $G$ and $G/H$ is cyclic then $H$ is finitely generated.
A subgroup $C$ of an ordered group is \textit{convex} if $x \in C$, whenever $1 \leq x \leq c$ with $c \in C$.

A \textit{relatively convex} subgroup of an O-group $G$ is a subgroup convex under some order on $G$.

The quotient $G/N$ of an O-group $G$ is an O-group if and only if $N$ is relatively convex.

If $C$, $D$ are convex subgroups of an ordered group $G$, $C < D$ and there is not a convex subgroup $H$ of $G$ such hat $C < H < D$, we say that $C \mapsto D$ is a \textit{convex jump} in $G$. 
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Lemma (3)

A convex subgroup of an ordered $k$-Engel group $(G, \leq)$ is normal in $G$.

Proof.

- Let $C$ be a convex subgroup of $G$, $g \in G$.
- The subgroup $g^{-1}Cg$ is also convex. For:
  \[1 \leq a \leq g^{-1}bg, \ b \in C \implies 1 \leq gag^{-1} \leq b \in C \implies gag^{-1} \in C \implies a \in g^{-1}Cg.\]
- Either $g^{-1}Cg \subseteq C$ or $C \subseteq g^{-1}Cg$.
  Assume w.l.o.g. $C \subseteq g^{-1}Cg$. Then $C \subseteq g^{-i}Cg^i$, for any $i > 0$ and $g^{-i}Cg^i \subseteq C$, for any $i < 0$.
- Assume $C \subseteq g^{-1}Cg$ and let $c \in C$ such that $g^{-1}cg \notin C$.
- By Lemma 1, $< c > < g > \subseteq g^{-s}Cg^s$, for some $s > 0$.
- Therefore $g^{-(s+1)}cg^{s+1} \in g^{-s}Cg^s$, from which $g^{-1}cg \in C$, a contradiction.
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Obviously an orderable group is an \(l\)-group. In 1988 N. Ya Medvedev proved the following

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\textit{Every k-Engel l-group is residually orderable.}
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$$x \leq y \implies xb \leq yb, \forall x, y, b \in G.$$ 

If $(G, \leq)$ is a partially right-ordered group and the order $\leq$ is a total order in $G$, we say that $(G, \leq)$ is a **right-ordered group**. 

$G$ is called a **right-orderable group** (an RO-group) if there exists a total order $\leq$ such that $(G, \leq)$ is a right-ordered group. 

Obviously an O-group is an RO-group and it is possible to prove that any lattice-orderable group is a subgroup of a right-orderable group. So is natural to ask:

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Is every RO $k$-Engel group nilpotent?

If $G$ is an RO $k$-Engel group, in order to show that $G$ is nilpotent it would be sufficient to prove that $G$ is locally indicable, i.e. every non trivial finitely generated subgroup of $G$ has an infinite cyclic factor group.

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Obviously an O-group is an RO-group and it is possible to prove that any lattice-orderable group is a subgroup of a right-orderable group. So is natural to ask:

Question

Is every RO $k$-Engel group nilpotent?

If $G$ is an RO $k$-Engel group, in order to show that $G$ is nilpotent it would be sufficient to prove that $G$ is locally indicable, i.e. every non trivial finitely generated subgroup of $G$ has an infinite cyclic factor group.

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Is every RO $k$-Engel group locally indicable?
If $\leq$ is a partial order in $G$, then $(G, \leq)$ is called a **partially right-ordered group** if,

$$x \leq y \implies xb \leq yb, \forall x, y, b \in G.$$ 

If $(G, \leq)$ is a partially right-ordered group and the order $\leq$ is a total order in $G$, we say that $(G, \leq)$ is a **right-ordered group**.

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A right partial order on the group $G$ is said to be archimedean if, for any $a, b \in G, a > 1, b > 1$, there exists a positive integer $n$ such that $b < a^n$. By a result of Hölder an order on $G$ is archimedean if and only if $G$ is order-isomorphic to a subgroup of the additive group of the real numbers under the natural order.

A right order on a group $G$ is a **Conrad order** if $C \trianglelefteq D$ and $D/C$ is archimedean for every convex jump $C \hookrightarrow D$.

**Theorem (Conrad)**

A group is locally indicable if and only if it has a right Conrad order.

**Theorem**

Every Conrad right-ordered $k$-Engel group is nilpotent of class bounded by a function of $k$. 
A right partial order on the group $G$ is said to be *archimedean* if, for any $a, b \in G, a > 1, b > 1$, there exists a positive integer $n$ such that $b < a^n$. By a result of Hölder an order on $G$ is archimedean if and only if $G$ is order-isomorphism to a subgroup of the additive group of the real numbers under the natural order.

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**Theorem (P.L, M. Maj, A. H. Rhemtulla)**

If an RO-group \((G, \leq)\) has no non-abelian free subsemigroups, then \(\leq\) is a Conrad order.

Therefore we get the following Question

Is it true that every RO \(k\)-Engel has no free non-abelian subsemigroups?

This is an old question for general \(k\)-Engel groups (see The Kourovka Notebook, Problem 2.82)

P. L. and M. Maj proved in 1997 that a right orderable 4-Engel group satisfies a non-trivial semigroup identity, hence it is nilpotent of bounded class. More generally G. Traustason proved in 1999 that any 4-Engel group satisfies a non-trivial semigroup identity and in 2005 G. Havas and M. Vaughan-Lee proved that 4-Engel groups are locally nilpotent. They are also Fitting groups by a result of G. Traustason.
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Notice that if we only assume that the ordered group $G$ is an Engel group it is not true that $G$ is necessarily nilpotent, as the following example shows.

**Example**

Let $A$ be an associative algebra over a field $K$. An element $a \in A$ is called *nilpotent* if $a^n = 0$ for some positive integer depending on $a$. If all elements of $A$ are nilpotent the $A$ is called a *nil-algebra*. The algebra $A$ is called *nilpotent* if there exists a positive integer $n$ such that $a_1 a_2 \cdots a_n = 0$, for any $a_1, \cdots, a_n \in A$. Obviously every nilpotent algebra is a nil-algebra, the converse is not true. Let $A$ be an associative algebra with a unit element $1$ and $B$ a nil-subalgebra of $A$. The elements of the form $1 + u$, $u \in B$, with the product of $A$ form a group $G(B)$. It is easy to prove that this group is nilpotent if $B$ is nilpotent.

E. S. Golod constructed in 1966, for any field $K$ and any integer $d \geq 3$, a non-nilpotent $d$-generated associative algebra $F$ such that every $(d - 1)$-generated subalgebra is nilpotent. The group $G(F)$ is a non-nilpotent Engel group. V. V. Bludov, A.M.W. Glass and A. H. Rhemtulla noticed in 2005 that if $K$ is of characteristic 0, then the group $G(F)$ is residually-(torsion-free nilpotent), thus it is also orderable.
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**Question (V. V. Bludov, problem 16.15 in The Kourovka Notebook)**

*Does the set of left Engel elements of an ordered group form a subgroup?*

It is also open the following

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*Is every lattice ordered Engel group residually orderable?*

Recently V. V. Bludov, A.M.W. Glass and A. H. Rhemtulla proved in 2005 the following interesting results

**Theorem**

*If an orderable group is generated by left Engel elements, then every convex jump is central.*

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*If an orderable group G is an Engel group, then every two-generated subgroup of G has all convex jumps central.*
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Engel conditions in combinatorial problems

Let $X$ be a class of groups. Given a group $G$, let $\Gamma_{X^\circ}(G)$ be the simple graph whose vertices are the elements of $G$, and different vertices $x$ and $y$ are connected by an edge if the subgroup $\langle x, y \rangle$ belongs to the class $X$. The group $G$ is said to be an $X^\circ$–group if the graph $\Gamma_{X^\circ}(G)$ has no infinite totally disconnected subgraphs, i.e. in any infinite subset of $G$ there exist different elements $x, y$ such that $\langle x, y \rangle \in X$.

If the set $S(X)$ is a subgroup of $G$ of finite index, where $S(X)$ consists of all elements $a \in G$ such that, for any $g \in G$, $\langle a, g \rangle \in X$, then it is easy to show that $G$ is a $X^\circ$-group.

For example this is true if $X = A$, where $A$ denotes the class of all abelian groups: here $S(A) = Z(G)$; conversely B. H. Neumann, answering to a question posed by P. Erdös, proved he following result.

**Theorem (B. H. Neumann, 1976)**

$G$ is an $A^\circ$-group if and only if $G/Z(G)$ is finite.

The proof uses Ramsey’s theorem.
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If the set $S(X)$ is a subgroup of $G$ of finite index, where $S(X)$ consists of all elements $a \in G$ such that, for any $g \in G$, $\langle a, g \rangle \in X$, then it is easy to show that $G$ is a $X^\circ$-group.

For example this is true if $X = A$, where $A$ denotes the class of all abelian groups: here $S(A) = Z(G)$; conversely B. H. Neumann, answering to a question posed by P. Erdős, proved he following result

**Theorem (B. H. Neumann, 1976)**

$G$ is an $A^\circ$-group if and only if $G/Z(G)$ is finite.

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For any prime $p \geq 5$, M.R. Vaughan-Lee and J. Wiegold constructed in 1981 a countable locally finite group of exponent $p$ which is perfect, and such that each of its 2-generator subgroups is nilpotent of bounded class. Hence the result of the previous theorem does not hold in general, even if we assume there is a bound for the nilpotence class of all 2-generated subgroups.
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Let $G$ be a finitely generated locally graded $N_k^\circ$-group. Then there is a positive integer $c$ depending only on $k$ such that $G/Z_c(G)$ is finite.

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**Theorem (P. L., 2001)**

Let $G$ be a finitely generated locally graded $E_k^\circ$-group. Then $G$ is finite-by-$(k$-Engel) (in particular it is a finite extension of a $k$-Engel group).

**Proof.**

- First we show that if $G$ is a torsion-free nilpotent group such that in every infinite subset $X$ of $G$ there exist two elements $x, y$ s.t. $[x, ky] = 1 = [y, kx]$, then $G$ is a $k$-Engel group.

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Suppose now that $X$ is a variety defined by the two-variables law $w(s, t) = 1$.
Given a group $G$, let $\Gamma_X^*(G)$ be the simple graph whose vertices are all elements of $G$, and different vertices $x$ and $y$ are connected by an edge if $w(x, y) = 1$.
The group $G$ is said to be an $X^\star$-group if the graph $\Gamma_X^*(G)$ has no infinite totally disconnected subgraphs.

Of course, every $X^\circ$-group is an $X^\star$-group.
If $A$ is the variety of abelian groups defined by the law $[x, y] = 1$, then obviously $A^\circ = A^\star$. 
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If $A$ is the variety of abelian groups defined by the law $[x, y] = 1$, then obviously $A^\circ = A^*$. 
It is much more difficult to deal with the class $E_k^\ast$, where $E_k$ is the variety of all $k$-Engel groups defined by the two-variables law $[x, ky] = 1$.

It is possible to show the following result:

**Theorem (A. Abdollahi, 2000)**

Let $G$ be a finitely generated soluble $E_k^\ast$-group. Then there is a positive integer $c$ depending only on $k$ such that $G/Z_c(G)$ is finite.

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**Theorem (C. Delizia, C. Nicotera, 2007)**

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**Proof.**

- First we show that $\langle x \rangle \langle y \rangle$ is finitely generated, for any $x, y \in G$. In fact, obviously we may assume that $y$ has infinite order. Thus the set 

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is infinite.

There exist different integers $i, j > 1$ such that $[xy^i, xy^j, xy^j] = 1$. It easily follows that 

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$$\langle x y^n : n \geq 0 \rangle \leq \langle x y^n : |n| < \max \{i, j\} \rangle.$$
Now starting from the infinite set \( \{xy^i : i < 1\} \) and repeating the previous argument, we can prove that
\[
\langle x^{y^n} : n \leq 0 \rangle \leq \langle x^{y^n} : |n| < \max \{h, k\} \rangle
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for suitable integers \( h, k > 1 \). Therefore
\[
\langle x \rangle \langle y \rangle = \langle x^{y^n} : |n| < m \rangle
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for some positive integer \( m \).

- By Lemma 2 the derived subgroup \( G' \) is finitely generated and, by induction, \( \gamma_i(G) \) is finitely generated, for all \( i > 0 \).
- Let \( R \) be the finite residual of \( G \). Since \( G/\gamma_i(G) \) is nilpotent and finitely generated, then it is residually finite and \( R \subseteq \gamma_i(G) \), for any \( i \geq 1 \).
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Since $G/R$ is residually finite, we obtain $\gamma_3(G)/R$ finite and $R$ is finitely generated.

If $R = \{1\}$, we have done. Otherwise there exists a normal subgroup $S$ of $R$, $S \lhd R$ and of finite index in $R$. We can assume $S$ normal in $G$. Then $G/S$ is residually finite and $R \subseteq S$, a contradiction.

More generally, of course, we can formulate the following:

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*If $V$ is a variety defined by the law $w = 1$, is it true that $V^* \subseteq V^\circ$?*
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*Does the equality $V \cup F = V^\#$ hold for any variety $V$ and for any word $w$?*

It is known that there exist classes of groups for which the previous equality holds.

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Let $V$ be a variety of groups defined by the law $w = 1$. Then an infinite $V^\#$-group $G$ is a $V$-group in the following cases:

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