Class preserving automorphisms of finite p-groups: A survey

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Motivation

The story begins in 1911, with the following question of W. Burnside:

- Does there exist any finite group G such that G has a non-inner class preserving automorphism?

In 1913, Burnside himself gave an affirmative answer to this question. He constructed a group G of order p^6 isomorphic to the group W consisting of all 3×3 matrices

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For this group G he proved:

- $\mathrm{Inn}(G) < \mathrm{Aut}_c(G)$.
- $\mathrm{Aut}_c(G)$ is an elementary abelian p-group of order p^8.

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- In 1947 G. E. Wall constructed some more examples of finite groups G such that $\mathrm{Out}_c(G) \neq 1$.
- Interestingly his examples contain 2-group having class preserving outer automorphisms.
- The smallest of these groups is a group of order 32.
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- He used the concepts of c-closed and strongly c-closed subgroups along with cohomological techniques to explore many nice basic properties of $\mathrm{Aut}_c(G)$ and $\mathrm{Out}_c(G)$ for a given finite group G.
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Results of C. H. Sah

- For nilpotent group he proved

Theorem

Let G be a nilpotent group of class c. Then $\mathrm{Aut}_c(G)$ is a nilpotent group of class $c-1$.

- As an immediate consequence of this result we have the following corollary.

Corollary

Let G be a nilpotent group of class c. Then $\mathrm{Out}_c(G)$ is a nilpotent group of class at most $c-1$.

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Let G be a finite solvable group. Then $\mathrm{Aut}_c(G)$ is solvable.

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Let G be a group admitting a composition series. Suppose that for each composition factor F of G the group $\mathrm{Aut}(F)/\mathrm{Inn}(F)$ is solvable. Then $\mathrm{Out}_c(G)$ is solvable.

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- Since every finite group admits a composition series and the Schreier's conjecture, i.e., the group of outer automorphisms of a finite simple group is solvable, holds true for all finite simple groups, the following result is a consequence of the above theorem.

Corollary

Let G be a finite group. Then $\mathrm{Out}_c(G)$ *is solvable.*

- Sah also constructed examples of finite groups G of order p^{5n}, $n \geq 3$, such that $\mathrm{Out}_c(G)$ is non-abelian. Thus these examples contradict the intuitions of W. Burnside that $\mathrm{Out}_c(G)$ should be abelian for all finite groups G.

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- In particular, $\mathrm{Out}_c(G) \neq 1$ for all of these group G. His groups are p-groups of nilpotency class 2.
- These groups also satisfy the property that $x\gamma_2(G) = x^G$ for all $x \in G - \gamma_2(G)$.

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- Continuing in this direction, I. Malinowska in 1992 constructed finite p-groups G of nilpotency class 3 and order p^6 such that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$.
- In the same paper she also constructed p-groups G of nilpotency class r, for any prime $p > 5$ and any integer $r > 2$ such that $\mathrm{Out}_c(G) \neq 1$.
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Isolcinism

- The following concept of isoclinism is due to P. Hall.
- Let X be a finite group and $\bar{X} = X/Z(X)$. Then commutation in X gives a well defined map $a_X : \bar{X} \times \bar{X} \mapsto \gamma_2(X)$ such that $a_X(xZ(X), yZ(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups G and H are called *isoclinic* if there exists an isomorphism ϕ of the factor group $\bar{G} = G/Z(G)$ onto $\bar{H} = H/Z(H)$, and an isomorphism θ of the subgroup $\gamma_2(G)$ onto $\gamma_2(H)$ such that the following diagram is commutative

$$\begin{array}{ccc} \bar{G} \times \bar{G} & \xrightarrow{a_G} & \gamma_2(G) \\ {\scriptstyle \phi\times\phi}\downarrow & & \downarrow{\scriptstyle \theta} \\ \bar{H} \times \bar{H} & \xrightarrow{a_H} & \gamma_2(H). \end{array}$$

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Isoclinism

- The resulting pair (ϕ, θ) is called an *isoclinism* of G onto H.
- Notice that isoclinism is an equivalence relation among finite groups.
- Each isoclinism class has a subgroup G such that $Z(G) \leq \gamma_2(G)$, which is called a stem group of the family.
- Any two groups in the same isoclinism family have the same commutator structure.

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Isoclinism and $\mathrm{Aut}_c(G)$

- As a consequence of the simple properties of isoclinism we have the following theorem, which essentially says that the group $\mathrm{Aut}_c(G)$ is independent of the choice of the group G in its isoclinism class.

Theorem

Let G and H be two finite non-abelian isoclinic groups. Then $\mathrm{Aut}_c(G) \cong \mathrm{Aut}_c(H)$.

- Since every isoclinism class has a stem group, we readily get the following result.

Corollary

It is sufficient to study $\mathrm{Aut}_c(G)$ for all finite groups G such that $Z(G) \leq \gamma_2(G)$.

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Groups with small order

- In the next few slides we are going to talk about the class preserving automorphisms of groups of small order, i.e., at most p^5.
- Obviously $\mathrm{Out}_c(G) = 1$ for all abelian groups.
- Let G be an extraspecial p-group. Then $\mathrm{Out}_c(G) = 1$.
- Thus $\mathrm{Out}_c(G) = 1$ for group of order p^3.
- Let G be a group of order p^4. Then $\mathrm{Out}_c(G) = 1$.

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Groups with small orders

Now we consider groups of order p^5, where p is an odd prime. Set

$$G_7 = \langle a, b, x, y, z \mid \mathcal{R}, [b, a] = 1 \rangle \text{ for } p \geq 3$$

and

$$G_{10} = \langle a, b, x, y, z \mid \mathcal{R}, [b, a] = x \rangle \text{ for } p \geq 5,$$

where $\mathcal{R}$ is defined as follows:

$$\begin{aligned} \mathcal{R} &= \{a^p = b^p = x^p = y^p = z^p = 1\} \cup \{[x, y] = [x, z] = [y, z] = 1\} \\ &\quad \cup \{x^b = xz, y^b = y, z^b = z\} \cup \{x^a = xy, y^a = yz, z^a = z\}. \end{aligned}$$

Groups of small orders

- For $p = 3$ define a group H of order 3^5 by

$$\begin{aligned} H &= \langle a, b, c \mid a^3 = b^9 = c^9 = 1, [b, c] = c^3, [a, c] = b^3, \\ &\quad [b, a] = c \rangle \\ &= \langle a \rangle \ltimes (\langle b \rangle \ltimes \langle c \rangle). \end{aligned}$$

- The group $\phi_7(1^5)$ in the isoclinism family (7) of R. James is isomorphic to G_7. The group $\phi_{10}(1^5)$ in the isoclinism family (10) is isomorphic to G_{10} for $p \geq 5$ and is isomorphic to H for $p = 3$.

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- The proof of the following theorem uses the classification of groups of order p^5 by R. James:

Theorem

Let G be a finite group of order p^5, where p is an odd prime. Then $\mathrm{Out}_c(G) \neq 1$ if and only if G is isoclinic to one of the groups G_7, G_{10} and H.

- This theorem, along with the examples of Wall proves that 4 is the smallest value of an integer n such that all the groups of order less than or equal to p^n has the property that $\mathrm{Out}_c(G) = 1$.

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Groups with large cyclic subgroups

Now we discuss on the class preserving automorphisms of finite p-groups of order p^n having cyclic subgroups of order p^{n-1} or p^{n-2}.

- We have the following result for p-groups with maximal cyclic subgroups:

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Let G be a finite p-group having a maximal cyclic subgroup. Then $\mathrm{Out}_c(G) = 1$.

- For odd primes p, we have a similar result for finite p-groups G having a cyclic subgroup of index p^2.

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- But the situation is different for $p = 2$. Let us set

$$\begin{aligned} G_1 &= \langle x, y, z \mid x^{2^{m-2}} = 1 = y^2 = z^2, yxy = x^{1+2^{m-3}}, zyz = y, \\ &\quad zxz = x^{-1+2^{m-3}} \rangle; \\ G_2 &= \langle x, y, z \mid x^{2^{m-2}} = 1 = y^2 = z^2, yxy = x^{1+2^{m-3}}, \\ &\quad zxz = x^{-1+2^{m-3}}, zyz = yx^{2^{m-3}} \rangle. \end{aligned}$$

- Then we get the following result:

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Let G be a group of order 2^n, $n > 4$, such that G has a cyclic normal subgroup of order 2^{n-2}, but does not have any element of order 2^{n-1}. Then $\mathrm{Out}_c(G) \neq 1$ if and only if G is isomorphic to G_1 or G_2.

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Groups of nilpotency class 2

- Let G be a finite nilpotent group of class 2 and $\phi \in \mathrm{Aut}_c(G)$.
- Then the map $g \mapsto g^{-1}\phi(g)$ is a homomorphism of G into $\gamma_2(G)$.
- This homomorphism sends $Z(G)$ to 1. So it induces a homomorphism $f_\phi : G/Z(G) \to \gamma_2(G)$, sending $gZ(G)$ to $g^{-1}\phi(g)$, for any $g \in G$.
- Notice that the map $\phi \mapsto f_\phi$ is a monomorphism of the group $\mathrm{Aut}_c(G)$ into $\mathrm{Hom}(G/Z(G), \gamma_2(G))$.

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- This homomorphism sends $Z(G)$ to 1. So it induces a homomorphism $f_\phi \colon G/Z(G) \to \gamma_2(G)$, sending $gZ(G)$ to $g^{-1}\phi(g)$, for any $g \in G$.
- Notice that the map $\phi \mapsto f_\phi$ is a monomorphism of the group $\mathrm{Aut}_c(G)$ into $\mathrm{Hom}(G/Z(G), \gamma_2(G))$.

Groups of nilpotency class 2

- Now any $\phi \in \mathrm{Aut}_c(G)$ sends any $g \in G$ to some $\phi(g) \in g^G$. Thus $f_\phi(gZ(G)) = g^{-1}\phi(g)$ lies in $g^{-1}g^G = [g, G]$.
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  by $\mathrm{Hom}_c(G/Z(G), \gamma_2(G))$.
- Then $f_\phi \in \mathrm{Hom}_c(G/Z(G), \gamma_2(G))$ for all $\phi \in \mathrm{Aut}_c(G)$.
- On the other hand, if $f \in \mathrm{Hom}_c(G/Z(G), \gamma_2(G))$, then the map sending any $g \in G$ to $gf(gZ(G))$ is an automorphism $\phi \in \mathrm{Aut}_c(G)$ such that $f_\phi = f$.

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Proposition

Let G be a finite nilpotent group of class 2. Then the above map $\phi \mapsto f_\phi$ is an isomorphism of the group $\mathrm{Aut}_c(G)$ onto $\mathrm{Hom}_c(G/\,Z(G), \gamma_2(G))$.

Theorem

Let G be a finite p-group of class 2. Let $\{x_1, x_2, \ldots, x_d\}$ be a minimal generating set for G such that $[x_i, G]$ is cyclic, $1 \leq i \leq d$. Then $\mathrm{Out}_c(G) = 1$.

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Camina p-groups

- Let G be a group and N be a non-trivial normal subgroup of G. The pair (G, N) is called Camina pair if $xN \subseteq x^G$ for all $x \in G - N$. G is called a Camina group is $(G, \gamma_2(G))$ is a Camina pair.
- Notice that extraspecial p-groups are Camina groups and $\mathrm{Out}_c(G) = 1$ for every extraspecial p-group G. The following result shows that these are the only Camina p-groups of class 2 for which $\mathrm{Out}_c(G) = 1$.

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Let G be a finite Camina p-group of nilpotency class 2. Then $|\mathrm{Aut}_c(G)| = |\gamma_2(G)|^d$, where d is the number of elements in a minimal generating set for G.

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- The following result deals with finite p-groups such that $(G, Z(G))$ is a Camina pair.

Theorem

Let G be a finite p-group of nilpotency class at least 3 such that $(G, Z(G))$ is a Camina pair and $|Z(G)| \geq p^2$. Then $|\mathrm{Aut}_c(G)| \geq |G|$.

- Since $(G, Z(G))$ is a Camina pair for every Camina p-group G, we have the following immediate corollary:

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An upper bound on $|\operatorname{Aut}_c(G)|$

- Let G be a finite p-group of order p^n and $\{x_1, \cdots, x_d\}$ be any minimal generating set for G.
- Let $\alpha \in \operatorname{Aut}_c(G)$. Since $\alpha(x_i) \in x_i^G$ for $1 \le i \le d$, there are at the most $|x_i^G|$ choices for the image of x_i under α.
- Thus it follows that

$$|\operatorname{Aut}_c(G)| \le \prod_{i=1}^{d} |x_i^G|. \tag{0.18}$$

- For $|\gamma_2(G)| = p^m$, we get

$$|\operatorname{Aut}_c(G)| \le p^{md} \le (p^m)^{n-m} = p^{m(n-m)}. \tag{0.19}$$

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Let G be a finite p-group. Equality holds in (0.19) if and only if G is either an abelian p-group or a non-abelian Camina special p-group.

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Let G be a non-trivial p-group having order p^n. Then

$$|\mathrm{Aut}_c(G)| \leq \begin{cases} p^{\frac{(n^2-4)}{4}}, & \text{if } n \text{ is even;} \\ p^{\frac{(n^2-1)}{4}}, & \text{if } n \text{ is odd.} \end{cases} \quad (0.22)$$

- Since the bound in this Theorem is attained by all non-abelian groups of order p^3 (n odd) and the group constructed by W. Burnside (n even), it follows that the bound in the theorem is optimal.

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- So a natural problem which arises here is the following
- **Problem.** Classify all finite p-groups G such that the above bound is attained by $|\mathrm{Aut}_c(G)|$.
- Consider the following group of order p^6, which is the group $\phi_{21}(1^6)$ in the isoclinism family (21) of R. James:

$$\begin{aligned} R \quad = \quad & \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, [\alpha, \alpha_1] = \beta_2, \\ & [\alpha, \alpha_2] = \beta_1^{\nu}, \alpha^p = \beta^p = \beta_i^p = 1, \alpha_1^p = \beta_1^{\binom{p}{3}}, \alpha_2^p = \beta_1^{-\binom{p}{3}}, \\ & i = 1, 2 \rangle, \end{aligned}$$

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- In the following theorem we classify all finite p-groups G for which the upper bound in the above theorem is achieved by $|\mathrm{Aut}_c(G)|$.

Theorem

Let G be a non-abelian finite p-group of order p^n. Then equality holds in (0.22) if and only if one of the following holds:

G is an extra-special p-group of order p^3;

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Open problems

Now we are going to state some problems which may be interesting for some of the audience here.

- We know that $\mathrm{Out}_c(G) = 1$ for all finite simple group. But the proof uses classification of finite groups. We think that there should be a direct proof.

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Let G be a finite simple group. Without using classification of finite simple groups prove that $\mathrm{Out}_c(G) = 1$.

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Let G be a finite p-group of nilpotency class 2. Find necessary and sufficient conditions on G such that $\mathrm{Out}_c(G) = 1$.

- Let G be a finite p-group of nilpotency class 2. Then $\mathrm{Aut}_c(G) \leq \mathrm{Autcent}(G)$.

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- As we mentioned in the introduction that H. Heineken constructed examples of finite p-groups G of nilpotency class 2 such that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$. This gives rise to the following natural problem.

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Let G be a finite p-group of nilpotency class $c > 2$. Give a sharp upper bound for the nilpotency class of $\mathrm{Out}_c(G)$.

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Classify all finite p-groups G such that $|\mathrm{Aut}_c(G)| = |\gamma_2(G)|^d$*, where d is the number of elements in a minimal generating set for G.*

- Notice that the Problem 7 has a solution when $\gamma_2(G) = \Phi(G)$.
- Let G be a finite p-group such that $Z(G) \le \gamma_2(G)$, then it follows that G is purely non-abelian and therefore $\mathrm{Autcent}(G)$ is a p-group.

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Let G be a finite p-group such that $\mathrm{Out}_c(G) \neq 1$ *and* $Z(G) \le \gamma_2(G)$*. Find a sharp lower bound for* $|\mathrm{Aut}_c(G)\,\mathrm{Autcent}(G)|$.

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Oh well.
My motto for writing mathematics is
“through errors to the truth.”

Everett C. Dade