Class preserving automorphisms of finite $p$-groups: A survey

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The story begins in 1911, with the following question of W. Burnside:

Does there exist any finite group $G$ such that $G$ has a non-inner class preserving automorphism?

In 1913, Burnside himself gave an affirmative answer to this question. He constructed a group $G$ of order $p^6$ isomorphic to the group $W$ consisting of all $3 \times 3$ matrices

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with $x, y, z$ in the field $\mathbb{F}_{p^2}$ of $p^2$ elements, where $p$ is an odd prime.
Motivation

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with $x, y, z$ in the field $\mathbb{F}_{p^2}$ of $p^2$ elements, where $p$ is an odd prime.
For this group $G$ he proved:

- $\text{Inn}(G) \leq \text{Aut}_c(G)$.

- $\text{Aut}_c(G)$ is an elementary abelian $p$-group of order $p^8$. 
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In 1947 G. E. Wall constructed some more examples of finite groups $G$ such that $\text{Out}_c(G) \neq 1$.
Interestingly his examples contain 2-group having class preserving outer automorphisms.
The smallest of these groups is a group of order 32.
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He used the concepts of $c$-closed and strongly $c$-closed subgroups along with cohomological techniques to explore many nice basic properties of $\text{Aut}_c(G)$ and $\text{Out}_c(G)$ for a given finite group $G$.

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We mention a few results of Sah here.
For nilpotent group he proved

**Theorem**

*Let $G$ be a nilpotent group of class $c$. Then $\text{Aut}_c(G)$ is a nilpotent group of class $c - 1$.***

As an immediate consequence of this result we have the following corollary.

**Corollary**

*Let $G$ be a nilpotent group of class $c$. Then $\text{Out}_c(G)$ is a nilpotent group of class at most $c - 1$.***
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Let $G$ be a finite solvable group. Then $\text{Aut}_c(G)$ is solvable.

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Let $G$ be a group admitting a composition series. Suppose that for each composition factor $F$ of $G$ the group $\text{Aut}(F)/\text{Inn}(F)$ is solvable. Then $\text{Out}_c(G)$ is solvable.
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Since every finite group admits a composition series and the Schreier’s conjecture, i.e., the group of outer automorphisms of a finite simple group is solvable, holds true for all finite simple groups, the following result is a consequence of the above theorem.

**Corollary**

*Let $G$ be a finite group. Then $\text{Out}_c(G)$ is solvable.*

Sah also constructed examples of finite groups $G$ of order $p^{5n}$, $n \geq 3$, such that $\text{Out}_c(G)$ is non-abelian. Thus these examples contradict the intuitions of W. Burnside that $\text{Out}_c(G)$ should be abelian for all finite groups $G$. 
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After this in 1980 H. Heineken, on the way to produce examples of finite groups in which all normal subgroups are characteristic, constructed finite $p$-groups $G$ with $\text{Aut}(G) = \text{Aut}_c(G)$.

In particular, $\text{Out}_c(G) \neq 1$ for all of these group $G$. His groups are $p$-groups of nilpotency class 2.

These groups also satisfy the property that $x^{\gamma_2(G)} = x^G$ for all $x \in G - \gamma_2(G)$.
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Continuing in this direction, I. Malinowska in 1992 constructed finite $p$-groups $G$ of nilpotency class 3 and order $p^6$ such that $\text{Aut}(G) = \text{Aut}_c(G)$.

In the same paper she also constructed $p$-groups $G$ of nilpotency class $r$, for any prime $p > 5$ and any integer $r > 2$ such that $\text{Out}_c(G) \neq 1$.

In 1988 W. Feit and G. M. Seitz proved that $\text{Out}_c(G) = 1$ for all finite simple groups $G$. 
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The following concept of isoclinism is due to P. Hall.

Let $X$ be a finite group and $\bar{X} = X/\text{Z}(X)$. Then commutation in $X$ gives a well defined map $a_X : \bar{X} \times \bar{X} \rightarrow \gamma_2(X)$ such that $a_X(x \text{Z}(X), y \text{Z}(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups $G$ and $H$ are called *isoclinic* if there exists an isomorphism $\phi$ of the factor group $\bar{G} = G/\text{Z}(G)$ onto $\bar{H} = H/\text{Z}(H)$, and an isomorphism $\theta$ of the subgroup $\gamma_2(G)$ onto $\gamma_2(H)$ such that the following diagram is commutative.

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\begin{array}{ccc}
\bar{G} \times \bar{G} & \xrightarrow{a_G} & \gamma_2(G) \\
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The resulting pair \((\phi, \theta)\) is called an *isoclinism* of \(G\) onto \(H\).

Notice that isoclinism is an equivalence relation among finite groups.

Each isoclinism class has a subgroup \(G\) such that 
\[ Z(G) \leq \gamma_2(G), \] 
which is called a stem group of the family.

Any two groups in the same isoclinism family have the same commutator structure.
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As a consequence of the simple properties of isoclinism we have the following theorem, which essentially says that the group $\text{Aut}_c(G)$ is independent of the choice of the group $G$ in its isoclinism class.

**Theorem**

Let $G$ and $H$ be two finite non-abelian isoclinic groups. Then $\text{Aut}_c(G) \cong \text{Aut}_c(H)$.

Since every isoclinism class has a stem group, we readily get the following result.

**Corollary**

It is sufficient to study $\text{Aut}_c(G)$ for all finite groups $G$ such that $Z(G) \leq \gamma_2(G)$.
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In the next few slides we are going to talk about the class preserving automorphisms of groups of small order, i.e., at most $p^5$.

- Obviously $\text{Out}_c(G) = 1$ for all abelian groups.
- Let $G$ be an extraspecial $p$-group. Then $\text{Out}_c(G) = 1$.
- Thus $\text{Out}_c(G) = 1$ for group of order $p^3$.
- Let $G$ be a group of order $p^4$. Then $\text{Out}_c(G) = 1$.
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Let $G$ be a group of order $p^4$. Then $\text{Out}_c(G) = 1$. 
Now we consider groups of order $p^5$, where $p$ is an odd prime. Set

\[ G_7 = \langle a, b, x, y, z \mid \mathcal{R}, [b, a] = 1 \rangle \text{ for } p \geq 3 \]

and

\[ G_{10} = \langle a, b, x, y, z \mid \mathcal{R}, [b, a] = x \rangle \text{ for } p \geq 5, \]

where $\mathcal{R}$ is defined as follows:

\[ \mathcal{R} = \{ a^p = b^p = x^p = y^p = z^p = 1 \} \cup \{ [x, y] = [x, z] = [y, z] = 1 \} \]

\[ \cup \{ x^b = xz, y^b = y, z^b = z \} \cup \{ x^a = xy, y^a = yz, z^a = z \}. \]
For $p = 3$ define a group $H$ of order $3^5$ by

$$H = \langle a, b, c \mid a^3 = b^9 = c^9 = 1, [b, c] = c^3, [a, c] = b^3, [b, a] = c \rangle = \langle a \rangle \ltimes (\langle b \rangle \ltimes \langle c \rangle).$$

The group $\phi_7(1^5)$ in the isoclinism family (7) of R. James is isomorphic to $G_7$. The group $\phi_{10}(1^5)$ in the isoclinism family (10) is isomorphic to $G_{10}$ for $p \geq 5$ and is isomorphic to $H$ for $p = 3$. 
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The proof of the following theorem uses the classification of groups of order $p^5$ by R. James:

**Theorem**

Let $G$ be a finite group of order $p^5$, where $p$ is an odd prime. Then $\text{Out}_c(G) \neq 1$ if and only if $G$ is isoclinic to one of the groups $G_7$, $G_{10}$ and $H$.

This theorem, along with the examples of Wall proves that 4 is the smallest value of an integer $n$ such that all the groups of order less than or equal to $p^n$ has the property that $\text{Out}_c(G) = 1$. 
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Now we discuss on the class preserving automorphisms of finite $p$-groups of order $p^n$ having cyclic subgroups of order $p^{n-1}$ or $p^{n-2}$.

- We have the following result for $p$-groups with maximal cyclic subgroups:

**Theorem**

Let $G$ be a finite $p$-group having a maximal cyclic subgroup. Then $\text{Out}_c(G) = 1$.

- For odd primes $p$, we have a similar result for finite $p$-groups $G$ having a cyclic subgroup of index $p^2$.

**Theorem**

Let $G$ be a group of order $p^n$ having a cyclic subgroup of order $p^{n-2}$, where $p$ is an odd prime. Then $\text{Out}_c(G) = 1$. 
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**Theorem**

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Groups with large cyclic subgroups

- But the situation is different for $p = 2$. Let us set

$$G_1 = \langle x, y, z | x^{2m-2} = 1 = y^2 = z^2, yxy = x^{1+2m-3}, zyz = y, zxz = x^{-1+2m-3} \rangle;$$

$$G_2 = \langle x, y, z | x^{2m-2} = 1 = y^2 = z^2, yxy = x^{1+2m-3}, zxz = x^{-1+2m-3}, zyz = yx^{2m-3} \rangle.$$

- Then we get the following result:

**Theorem**

Let $G$ be a group of order $2^n$, $n > 4$, such that $G$ has a cyclic normal subgroup of order $2^{n-2}$, but does not have any element of order $2^{n-1}$. Then $\Out_c(G) \neq 1$ if and only if $G$ is isomorphic to $G_1$ or $G_2$. 
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Let \( G \) be a group of order \( 2^n, n > 4 \), such that \( G \) has a cyclic normal subgroup of order \( 2^{n-2} \), but does not have any element of order \( 2^{n-1} \). Then \( \text{Out}_c(G) \neq 1 \) if and only if \( G \) is isomorphic to \( G_1 \) or \( G_2 \).
Let $G$ be a finite nilpotent group of class 2 and $\phi \in \text{Aut}_c(G)$. Then the map $g \mapsto g^{-1}\phi(g)$ is a homomorphism of $G$ into $\gamma_2(G)$. This homomorphism sends $Z(G)$ to 1. So it induces a homomorphism $f_\phi : G/Z(G) \to \gamma_2(G)$, sending $gZ(G)$ to $g^{-1}\phi(g)$, for any $g \in G$.

Notice that the map $\phi \mapsto f_\phi$ is a monomorphism of the group $\text{Aut}_c(G)$ into $\text{Hom}(G/Z(G), \gamma_2(G))$. 
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Let $G$ be a finite nilpotent group of class 2 and $\phi \in \text{Aut}_c(G)$.

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Denote

$$\{ f \in \text{Hom}(G/Z(G), \gamma_2(G)) \mid f(gZ(G)) \in [g, G], \text{ for all } g \in G \}$$

by $\text{Hom}_c(G/Z(G), \gamma_2(G))$.

Then $f_\phi \in \text{Hom}_c(G/Z(G), \gamma_2(G))$ for all $\phi \in \text{Aut}_c(G)$.

On the other hand, if $f \in \text{Hom}_c(G/Z(G), \gamma_2(G))$, then the map sending any $g \in G$ to $gf(gZ(G))$ is an automorphism $\phi \in \text{Aut}_c(G)$ such that $f_\phi = f$. 
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Groups of nilpotency class 2

Thus we have

**Proposition**

Let $G$ be a finite nilpotent group of class 2. Then the above map $\phi \mapsto f_\phi$ is an isomorphism of the group $\text{Aut}_c(G)$ onto $\text{Hom}_c(G/ Z(G), \gamma_2(G))$.

**Theorem**

Let $G$ be a finite $p$-group of class 2. Let $\{x_1, x_2, \ldots, x_d\}$ be a minimal generating set for $G$ such that $[x_i, G]$ is cyclic, $1 \leq i \leq d$. Then $\text{Out}_c(G) = 1$.

**Corollary**

Let $G$ be a finite $p$-group of class 2 such that $\gamma_2(G)$ is cyclic. Then $\text{Out}_c(G) = 1$.
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**Proposition**

*Let G be a finite nilpotent group of class 2. Then the above map \( \phi \mapsto f_\phi \) is an isomorphism of the group \( \text{Aut}_c(G) \) onto \( \text{Hom}_c(G/Z(G), \gamma_2(G)) \).*

**Theorem**

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**Proposition**

Let $G$ be a finite nilpotent group of class 2. Then the above map $\phi \mapsto f_\phi$ is an isomorphism of the group $\text{Aut}_c(G)$ onto $\text{Hom}_c(G/\text{Z}(G), \gamma_2(G))$.

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**Corollary**

Let $G$ be a finite $p$-group of class 2 such that $\gamma_2(G)$ is cyclic. Then $\text{Out}_c(G) = 1$. 
Let $G$ be a group and $N$ be a non-trivial normal subgroup of $G$. The pair $(G, N)$ is called Camina pair if $xN \subseteq x^G$ for all $x \in G - N$. $G$ is called a Camina group if $(G, \gamma_2(G))$ is a Camina pair.

Notice that extraspecial $p$-groups are Camina groups and $\text{Out}_c(G) = 1$ for every extraspecial $p$-group $G$. The following result shows that these are the only Camina $p$-groups of class 2 for which $\text{Out}_c(G) = 1$.

**Theorem**

Let $G$ be a finite Camina $p$-group of nilpotency class 2. Then $|\text{Aut}_c(G)| = |\gamma_2(G)|^d$, where $d$ is the number of elements in a minimal generating set for $G$. 
Let $G$ be a group and $N$ be a non-trivial normal subgroup of $G$. The pair $(G, N)$ is called Camina pair if $xN \subseteq x^G$ for all $x \in G - N$. $G$ is called a Camina group if $(G, \gamma_2(G))$ is a Camina pair.

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Class preserving automorphisms of finite $p$-groups: A survey
The following result deals with finite $p$-groups such that $(G, Z(G))$ is a Camina pair.

**Theorem**

Let $G$ be a finite $p$-group of nilpotency class at least 3 such that $(G, Z(G))$ is a Camina pair and $|Z(G)| \geq p^2$. Then $|\text{Aut}_c(G)| \geq |G|$.

Since $(G, Z(G))$ is a Camina pair for every Camina $p$-group $G$, we have the following immediate corollary:

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Let $G$ be a finite Camina $p$-group of nilpotency class 3 and $|Z(G)| \geq p^2$. Then $|\text{Aut}_c(G)| \geq |G|$.
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Let $\alpha \in \text{Aut}_c(G)$. Since $\alpha(x_i) \in x_i^G$ for $1 \leq i \leq d$, there are at the most $|x_i^G|$ choices for the image of $x_i$ under $\alpha$.

Thus it follows that

$$|\text{Aut}_c(G)| \leq \prod_{i=1}^{d} |x_i^G|. \quad (0.18)$$

For $|\gamma_2(G)| = p^m$, we get

$$|\text{Aut}_c(G)| \leq p^{md} \leq (p^m)^{n-m} = p^{m(n-m)}. \quad (0.19)$$
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Let $G$ be a finite $p$-group. Equality holds in (0.19) if and only if $G$ is either an abelian $p$-group or a non-abelian Camina special $p$-group.

Let $G$ be a non-trivial $p$-group having order $p^n$. Then

$$|\text{Aut}_c(G)| \leq \begin{cases} 
  p^{\frac{(n^2-4)}{4}}, & \text{if } n \text{ is even;} \\
  p^{\frac{(n^2-1)}{4}}, & \text{if } n \text{ is odd.}
\end{cases} \quad (0.22)$$

Since the bound in this Theorem is attained by all non-abelian groups of order $p^3$ ($n$ odd) and the group constructed by W. Burnside ($n$ even), it follows that the bound in the theorem is optimal.
Let $G$ be a finite $p$-group. Equality holds in (0.19) if and only if $G$ is either an abelian $p$-group or a non-abelian Camina special $p$-group.

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An upper bound on $|\text{Aut}_c(G)|$

Theorem

Let $G$ be a finite $p$-group. Equality holds in (0.19) if and only if $G$ is either an abelian $p$-group or a non-abelian Camina special $p$-group.

Theorem

Let $G$ be a non-trivial $p$-group having order $p^n$. Then

$$|\text{Aut}_c(G)| \leq \begin{cases} \frac{p^{(n^2-4)/4}}{4}, & \text{if } n \text{ is even;} \\ \frac{p^{(n^2-1)/4}}{4}, & \text{if } n \text{ is odd.} \end{cases}$$ (0.22)

Since the bound in this Theorem is attained by all non-abelian groups of order $p^3$ ($n$ odd) and the group constructed by W. Burnside ($n$ even), it follows that the bound in the theorem is optimal.
So a natural problem which arises here is the following

**Problem.** Classify all finite $p$-groups $G$ such that the above bound is attained by $|\text{Aut}_c(G)|$.

Consider the following group of order $p^6$, which is the group $\phi_{21}(1^6)$ in the isoclinism family (21) of R. James:

$$R = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^\nu, \alpha^p = \beta^p = \beta_i^p = 1, \alpha_1^p = \beta_1^{(p)}, \alpha_2^p = \beta_1^{-(p)}, i = 1, 2 \rangle,$$

where $\nu$ is the smallest positive integer which is a non-quadratic residue mod $p$ and $\beta_1$ and $\beta_2$ are central elements.
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In the following theorem we classify all finite $p$-groups $G$ for which the upper bound in the above theorem is achieved by $|\text{Aut}_c(G)|$.

**Theorem**

Let $G$ be a non-abelian finite $p$-group of order $p^n$. Then equality holds in (0.22) if and only if one of the following holds:

- $G$ is an extra-special $p$-group of order $p^3$;
- $G$ is a group of nilpotency class 3 and order $p^4$;
- $G$ is a Camina special $p$-group isoclinic to the group $W$ and $|G| = p^6$;
- $G$ is isoclinic to $R$ and $|G| = p^6$. 
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Open problems

Now we are going to state some problems which may be interesting for some of the audience here.

- We know that $\text{Out}_c(G) = 1$ for all finite simple group. But the proof uses classification of finite groups. We think that there should be a direct proof.

**Problem**

Let $G$ be a finite simple group. Without using classification of finite simple groups prove that $\text{Out}_c(G) = 1$.

**Problem**

Let $G$ be a finite $p$-group of nilpotency class 2. Find necessary and sufficient conditions on $G$ such that $\text{Out}_c(G) = 1$.

- Let $G$ be a finite $p$-group of nilpotency class 2. Then $\text{Aut}_c(G) \leq \text{Autcent}(G)$. 
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**Problem**

Let $G$ be a finite $p$-group of nilpotency class $2$. Find necessary and sufficient conditions on $G$ such that $\text{Out}_c(G) = 1$.

- Let $G$ be a finite $p$-group of nilpotency class $2$. Then $\text{Aut}_c(G) \leq \text{Autcent}(G)$.
As we mentioned in the introduction that H. Heineken constructed examples of finite $p$-groups $G$ of nilpotency class 2 such that $\text{Aut}(G) = \text{Aut}_c(G)$. This gives rise to the following natural problem.

The following problem arises from the work of Sah, which is also given in Malinowska’s survey as Question 12.
Open problems

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Let \( G \) be a finite \( p \)-group of nilpotency class \( c > 2 \). Give a sharp upper bound for the nilpotency class of \( \text{Out}_c(G) \).

- Let \( G \) be a finite \( p \)-group with a minimal generating set \( \{x_1, x_2, \cdots, x_d\} \). Then \( |\text{Aut}_c(G)| \leq \prod_{i=1}^d |x_i^G| \).

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- Since \( |x_i^G| \leq |\gamma_2(G)| \), we can even formulate the following particular case of the preceding problem.
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Problem

Classify all finite $p$-group $G$ with a minimal generating set $\{x_1, x_2, \cdots, x_d\}$ such that $|\text{Aut}_c(G)| = \prod_{i=1}^{d} |x_i^G|$.

- Since $|x_i^G| \leq |\gamma_2(G)|$, we can even formulate the following particular case of the preceding problem.
Let $G$ be a finite $p$-group of nilpotency class $c > 2$. Give a sharp upper bound for the nilpotency class of $\text{Out}_c(G)$.

Let $G$ be a finite $p$-group with a minimal generating set $\{x_1, x_2, \cdots, x_d\}$. Then $|\text{Aut}_c(G)| \leq \prod_{i=1}^{d} |x^G|$.

Classify all finite $p$-group $G$ with a minimal generating set $\{x_1, x_2, \cdots, x_d\}$ such that $|\text{Aut}_c(G)| = \prod_{i=1}^{d} |x^G|$.

Since $|x^G| \leq |\gamma_2(G)|$, we can even formulate the following particular case of the preceding problem.
Open problems

Problem

Classify all finite $p$-groups $G$ such that $|\text{Aut}_c(G)| = |\gamma_2(G)|^d$, where $d$ is the number of elements in a minimal generating set for $G$.

- Notice that the Problem 7 has a solution when $\gamma_2(G) = \Phi(G)$.
- Let $G$ be a finite $p$-group such that $Z(G) \leq \gamma_2(G)$, then it follows that $G$ is purely non-abelian and therefore $\text{Aut}_{cent}(G)$ is a $p$-group.

Problem

Let $G$ be a finite $p$-group such that $\text{Out}_c(G) \neq 1$ and $Z(G) \leq \gamma_2(G)$. Find a sharp lower bound for $|\text{Aut}_c(G) \text{Aut}_{cent}(G)|$. 
Open problems

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-Manoj Kumar Yadav

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Problem

Let $G$ be a finite $p$-group such that $\text{Out}_c(G) \neq 1$ and $Z(G) \leq \gamma_2(G)$. Find a sharp lower bound for $|\text{Aut}_c(G)\text{Autcent}(G)|$. 
Open problems

Problem

Classify all finite \( p \)-groups \( G \) such that \( |\text{Aut}_c(G)| = |\gamma_2(G)|^d \), where \( d \) is the number of elements in a minimal generating set for \( G \).

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- Let \( G \) be a finite \( p \)-group such that \( Z(G) \leq \gamma_2(G) \), then it follows that \( G \) is purely non-abelian and therefore \( \text{Aut}_{\text{cent}}(G) \) is a \( p \)-group.

Problem

Let \( G \) be a finite \( p \)-group such that \( \text{Out}_c(G) \neq 1 \) and \( Z(G) \leq \gamma_2(G) \). Find a sharp lower bound for \( |\text{Aut}_c(G) \text{Aut}_{\text{cent}}(G)| \).
Oh well.
My motto for writing mathematics is “through errors to the truth.”

Everett C. Dade