Free groups in the variety generated by a finite group

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Introduction

Bernhard Hermann Neumann

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A. H. Rhemtulla — coauthored with Chander Kanta Gupta
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I met BHN in 1993 in Galway
at Groups St. Andrews 1993

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Question

What is the order of $F_2(var(S_3))$*?*

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Problem

Choose random element of $F_2(var(S_3))$.

History

1931 Doctorate in Berlin "Herr Doktor"
1933 August, moved to Cambridge
1933 – 1935 PhD student under Philip Hall

Result

"Identical relations in groups I." Matematische Annalen 114(1937), 506–525.

Theorem

The free groups of finite rank of a variety generated by a finite group G are finite. In fact for $n \geq 1$ the order $|F_n(var(G))|$ divides $|G|^{|G|^n}$.

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If $G = \{1, g_2, g_3, \ldots, g_n\}$ then $F_2(var(G))$ is generated in direct product $G^{|G|^2}$ by

$$h_1 = (\underbrace{1, g_2, \ldots, g_n}, \underbrace{1, g_2, \ldots, g_n}, \ldots\ldots, \underbrace{1, g_2, \ldots, g_n})$$

$$h_2 = (\underbrace{1, 1, \ldots, 1}, \underbrace{g_2, g_2, \ldots, g_2}, \ldots\ldots, \underbrace{g_n, g_n, \ldots, g_n})$$

Result

$$v(h_1, h_2) =$$
$$(v(1,1), v(g_2,1), \ldots, v(g_n,1), v(1,g_2), v(g_2,g_2), \ldots,$$
$$v(g_n,g_2), \ldots\ldots, v(g_n,g_n))$$

so $v(h_1, h_2) = 1$ if and only if $v(g, g') = 1$ for all $g, g' \in G$.

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$$h_1 = ((1,2),(1,2),(1,2,3),(1,2,3),(1,2,3),(1,2,3),1)$$

$$h_2 = ((2,3),(1,2,3),(1,2),(1,2,3),(1,3,2),1,(1,2,3))$$

generate $F_2(var(S_3))$ in $S_3 \times S_3 \times S_3 \times A_3 \times A_3 \times A_3 \times A_3$,

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We note

$$972 = 6^2 \cdot 3^3 = 2^2 \cdot 3^5$$

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Since $D_3 \simeq S_3$, he considered $|F_k(var(D_m))|$ where D_m is dihedral group of order $2m$, $m \geq 2$.

$$D_m =< a, b | a^2 = b^m = (ab)^2 = 1 >$$

Metabelian groups

Theorem

If m is odd, $F_k(var(D_m))$ has order dividing $2^k \cdot m^{(k-1)2^k+1}$.
If m is even, $F_k(var(D_m))$ has order dividing $2^k \cdot (m/2)^{(k-1)2^k+1}$.
If $m = 2$, $F_k(var(D_m)) = (\mathbb{Z}_2)^k$.

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If p is an odd prime, $|F_2(var(D_p))| = 4 \cdot p^5$.

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L. G. Kovács "Free groups in a dihedral variety" Proc. Royal Irish Acad. 89A, No 1(1989), 115–117.

Kovacs found a formula for the orders of the finite rank free groups of the variety generated by an arbitrary finite dihedral group.

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Theorem

If D is a dihedral group of order $2^{d+1} \cdot e$ (with e odd), then $F_r(var(D))$ has order $2^{r+s} \cdot e^{r'}$ where $r' = (r+1)2^r + 1$ and

$$s = \sum_{t=2}^{d} (d+1-t)(t-1)\binom{r+1}{t}$$

The relevant conventions are that $s = 0$ when $d < 2$ and the binomial coefficient vanishes when $t > r+1$.

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For example:
$var(\mathbb{Q}_8) = var(D_4) = \mathfrak{A}_2\mathfrak{A}_2 \wedge \mathfrak{N}_2$.

Symmetric polynomials

Symmetric polynomials in three variables:
$x + y + z$, $xy + xz + yz$, xyz — elementary symmetric polynomials

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Restrict our considerations to addition only (abelian groups).
So general 'polynomial' has a form:

$$\underbrace{x+x+\ldots+x}+\underbrace{y+y+\ldots+y}+\underbrace{z+z+\ldots+z}=k_1x+k_2y+k_3z$$

where k_1, k_2, k_3 are integers.

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Such polynomial is symmetric if and only if $k_1=k_2=k_3$,
i.e. has a form $k(x+y+z)$ for some integer k.

Symmetric words

Definition

An n-ary word v is called n-symmetric word for a group G if

$$v(g_1, g_2, \ldots, g_n) = v(g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(n)})$$

for all $g_1, \ldots, g_n \in G$ and all permutations $\sigma \in S_n$.

Symmetric words

G - the group
$var(G)$ - the variety generated by G.
$F_n(var(G))$ - the free group in $var(G)$ generated by $x_1, \dots, x_n$.

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$S^{(n)}(G)$ - the group of fixed points of A, the group of n-symmetric words for G.

Symmetric words

1) $G = F_k$ - free of rank k
$S^{(n)}(G)$ is trivial for $n > 1$ and infinite cyclic for $n = 1$

2) G - abelian
$v = x_1^{a_1} \dots x_n^{a_n}$
so $v \in S^{(n)}(G)$ iff $v = x_1^a \dots x_n^a = (x_1 \dots x_n)^a$

Symmetric words

Known results:
for all n
— nilpotent of class ≤ 3
— free nilpotent of class 4.
— nilpotent of class 4 (2-isolated)
$n = 2, 3$
— free metabelian
— free metabelian $+$ nilpotency class c
$n = 2$
— free soluble of class c
— free nilpotent class c-by-abelian
— free centre-by-metabelian
E. Płonka, S. Krstic, O. Macedonska, W. Tomaszewski, C. K. Gupta, WH,
...

Symmetric words in finite groups

G — finite $\Rightarrow S^{(n)}(G)$ — finite

G — of finite exponent $\Rightarrow S^{(n)}(G)$ — nontrivial

$(g_1 g_2^{-1})^k = 1$

If $k = 2s$ then $(g_1 g_2^{-1})^s = (g_2 g_1^{-1})^s$.
If $k = 2s + 1$ then $(g_1 g_2^{-1})^s g_1 = (g_2 g_1^{-1})^s g_2$.

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My financial resources = A pounds

Symmetric words in finite groups

n	G_n	$\|G_n\|$	$\|G'_n\|$	$exp(G_n)$	$\|F_2(var(G_n))\|$	Al
1	$S_3 \simeq D_3$	6	3	6	$972 = 2^2 \cdot 3^5$	3^3
2	D_4	8	2	4	$32 = 2^5$	2
4	D_5	10	5	10	$12500 = 2^2 \cdot 5^5$	5^3
6	A_4	12	4	6	$9216 = 2^{10} \cdot 3^2$	2^8
7	$< 2, 2, 3 >$	12	3	12	$3888 = 2^4 \cdot 3^5$	3^3
8	D_7	14	7	14	$67228 = 2^2 \cdot 7^5$	7^3
11	D_8	16	4	8	$1024 = 2^{10}$	2^4
12	$< -2, 4 \| 2 >$	16	4	8	$1024 = 2^{10}$	2^4
13	$< 2, 2 \| 2 >$	16	2	8	$128 = 2^7$	2
17	$< 2, 2, 4 >$	16	4	8	$1024 = 2^{10}$	2^4
19	D_9	18	9	18	$236196 = 2^2 \cdot 3^{10}$	3^6
20	$((3, 3, 3; 2))$	18	9	6	$972 = 2^2 \cdot 3^5$	3^3
22	$F^{2,1,-1}$	20	5	20	$12207031250000 = 2^4 \cdot 5^{17}$	5^{15}
23	$< 2, 2, 5 >$	20	5	20	$50000 = 2^4 \cdot 5^5$	5^3
24		21	7	21	$2542277241 = 3^2 \cdot 7^{10}$	7^8

Symmetric words

n	G_n	$\|G_n\|$	$\|G'_n\|$	$exp(G_n)$	$\|F_2(var(G_n))\|$	AI
25	D_{11}	22	11	22	$644204 = 2^2 \cdot 11^5$	11^3
28	$C_3 \times D_4$	24	2	12	$288 = 2^5 \cdot 3^2$	2
30	$C_4 \times D_3$	24	3	12	$3888 = 2^4 \cdot 3^5$	3^3
32	D_{12}	24	6	12	$7776 = 2^5 \cdot 3^5$	$2 \cdot 3^3$
33	S_4	24	12	12	$815372976 = 2^{25} \cdot 3^5$	
34	$< 2,3,3 >$	24	8	12	$1179648 = 2^{17} \cdot 3^2$	
35	$(4,6\|2,2)$	24	6	12	$7776 = 2^5 \cdot 3^5$	$2 \cdot 3^3$
36	$< -2,2,3 >$	24	3	24	$15552 = 2^6 \cdot 3^5$	3^3
37	$< 2,2,6 >$	24	6	12	$7776 = 2^5 \cdot 3^5$	$2 \cdot 3^3$
38	D_{13}	26	13	26	$1485172 = 2^2 \cdot (13)^5$	13^3
39	$(3,3\|3,3)$	27	3	3	$27 = 3^3$	3
40		27	3	9	$243 = 3^5$	3
42	$< 2,2,7 >$	28	7	28	$268912 = 2^4 \cdot 7^5$	7^3
43	$C_3 \times D_5$	30	5	30	$112500 = 2^2 \cdot 3^2 \cdot 5^5$	5^3
44	$C_5 \times D_3$	30	3	30	$24300 = 2^2 \cdot 3^5 \cdot 5^2$	3^3
45	D_{15}	30	15	30	$3037500 = 2^2 \cdot 3^5 \cdot 5^5$	$3^3 \cdot 5^3$

Symmetric words in finite groups

1) $S^{(2)}(D_p) \simeq C_p \times D_p$ (p-prime) E. Płonka
$D_{2p} \simeq C_2 \times D_p$

2) $S^{(1)}(Q_8) = \{1, x, x^2, x^3\}$
$S^{(2)}(Q_8) = \{1, x^2y^2, [x, y], x^2y^2[x, y]\} \simeq V_4$ (Klein group)
$S^{(n)}(Q_8) \simeq S^{(2)}(Q_8)$

3) $S^{(2)}(A_4) \simeq (C_2)^6$

4) $S^{(2)}(C_3 \times D_4) \simeq C_2 \times C_2 \times C_3$

5) 4) $S^{(2)}(C_4 \times D_3) \simeq C_3 \times < 2, 2, 3 >$
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Application to infinite groups

Theorem

In a finitely generated group every subgroup of finite index contains a verbal (fully invariant) subgroup of finite index.

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Proof: Lemma Poncaré: In a group every subgroup of finite index contains a normal subgroup of finite index.

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Let $G = \alpha(F_k)$ be a presentation of G, $N \lhd G$, $[G : N] < \infty$, $S = \alpha^{-1}(N)$.

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The free group of the variety $var(F_k/S)$ is finite (BHN)

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Claim: S is of finite index in F_k
The free group of the variety $var(F_k/S)$ is finite (BHN)
F_k/S is a factor group of this free group.
Hence $F_k(var(F_k/S)) \simeq F_k/V$
V is a verbal subgroup of finite index in F_k and $V \subseteq S$.

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F_k/S is a factor group of this free group.
Hence $F_k(var(F_k/S)) \simeq F_k/V$
V is a verbal subgroup of finite index in F_k and $V \subseteq S$.
Then $\alpha(V)$ is a verbal (fully invariant) subgroup, $[G : \alpha(V)] < \infty$ and $\alpha(V) \subseteq \alpha(S) = N$.