

# FINITE GROUPS OF LIE TYPE AND THEIR REPRESENTATIONS – LECTURE IV

Gerhard Hiss

Lehrstuhl D für Mathematik  
RWTH Aachen University

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# CONTENTS

- 1 Harish-Chandra theory
- 2 Decomposition numbers
- 3 Unipotent Brauer characters
- 4  $(q-)$ Schur algebras

# HARISH-CHANDRA CLASSIFICATION: RECOLLECTION

Throughout this lecture, let  $k$  be an algebraically closed field of characteristic  $\ell \geq 0$ , and let  $G$  be a finite group.

Recall that Harish-Chandra theory yields the following classification for a finite group of Lie type of characteristic  $p \neq \ell$ .

## THEOREM (HARISH-CHANDRA, LUSZTIG, GECK-H.-MALLE)

$$\begin{array}{c}
 \{ V \mid V \text{ simple } kG\text{-module} \} / \text{isomorphism} \\
 \updownarrow \\
 \left\{ (L, M, \theta) \mid \begin{array}{l} L \text{ Levi subgroup of } G \\ M \text{ simple, cuspidal } kL\text{-module} \\ \theta \text{ irred. } k\text{-rep'n of } \mathcal{H}(L, M) \end{array} \right\} / \text{conjugacy}
 \end{array}$$

# PROBLEMS IN HARISH-CHANDRA THEORY

The above theorem leads to the three tasks:

- 1 Determine the **cuspidal pairs**  $(L, M)$ .
- 2 For each of these, “compute”  $\mathcal{H}(L, M)$ .
- 3 Classify irreducible  $k$ -representations of  $\mathcal{H}(L, M)$ .

State of the art in case  $\ell > 0$ :

- $\mathcal{H}(L, M)$  is an Iwahori-Hecke algebra corresponding to an “extended” Coxeter group (Howlett-Lehrer (1980), Geck-H.-Malle (1996)), namely  $W_G(L, M)$ ; parameters of  $\mathcal{H}(L, M)$  not known in general.
- $G = \mathrm{GL}_n(q)$ ; everything known (Dipper-James, 1980s)
- $G$  classical group,  $\ell$  “linear”; everything known (Gruber-H., 1997).
- In general, classification of cuspidal pairs open.

# EXAMPLE: $\mathrm{SO}_{2m+1}(q)$ (GECK-H.-MALLE (1996))

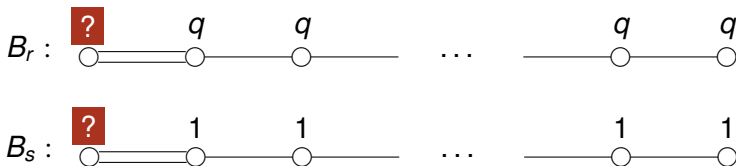
Let  $G = \mathrm{SO}_{2m+1}(q)$ , assume that  $\ell > m$ , and put  $e := \min\{i \mid \ell \text{ divides } q^i - 1\}$ , the order of  $q$  in  $\mathbb{F}_\ell^*$ .

Any Levi subgroup  $L$  of  $G$  containing a cuspidal unipotent (see below) module  $M$  is of the form

$$L = \mathrm{SO}_{2m'+1}(q) \times \mathrm{GL}_1(q)^r \times \mathrm{GL}_e(q)^s.$$

In this case  $W_G(L, M) \cong W(B_r) \times W(B_s)$ , where  $W(B_j)$  denotes a Weyl group of type  $B_j$ .

Moreover,  $\mathcal{H}(L, M) \cong \mathcal{H}_{k, \mathbf{q}}(B_r) \otimes \mathcal{H}_{k, \mathbf{q}}(B_s)$ , with  $\mathbf{q}$  as follows:



# BRAUER CHARACTERS

From now on suppose that  $\ell > 0$ .

Let  $\mathfrak{X}$  be a  $k$ -representation of  $G$  of degree  $d$ .

The character  $\chi_{\mathfrak{X}}$  of  $\mathfrak{X}$  as defined in Lecture 3 does not convey all the desired information, e.g.,

$\chi_{\mathfrak{X}}(1)$  only gives the degree  $d$  of  $\mathfrak{X}$  modulo  $\ell$ .

Instead one considers the **Brauer character**  $\varphi_{\mathfrak{X}}$  of  $\mathfrak{X}$ .

This is obtained by consistently lifting the eigenvalues of the matrices  $\mathfrak{X}(g)$  for  $g \in G_{\ell'}$  to characteristic 0. (Here,  $G_{\ell'}$  is the set of  $\ell$ -regular elements of  $G$ .)

Thus  $\varphi_{\mathfrak{X}} : G_{\ell'} \rightarrow K$ , where  $K$  is a suitable field with  $\text{char}(K) = 0$ , and  $\varphi_{\mathfrak{X}}(g) = \text{sum of the eigenvalues of } \mathfrak{X}(g)$  (viewed as elements of  $K$ ).

In particular,  $\varphi_{\mathfrak{X}}(1)$  equals the degree of  $\mathfrak{X}$ .

# THE BRAUER CHARACTER TABLE

If  $\chi$  is irreducible,  $\varphi_\chi$  is called an **irreducible Brauer character**.

Put  $\text{IBr}_\ell(G) :=$  set of irreducible Brauer characters of  $G$ ,  
 $\text{IBr}_\ell(G) = \{\varphi_1, \dots, \varphi_n\}$ .

(If  $\ell \nmid |G|$ , then  $\text{IBr}_\ell(G) = \text{Irr}(G)$ .)

Let  $g_1, \dots, g_n$  be representatives of the conjugacy classes contained in  $G_{\ell'}$  (same  $n$  as above!).

The square matrix

$$[\varphi_i(g_j)]_{1 \leq i, j \leq l}$$

is called the **Brauer character table** or  **$\ell$ -modular character table** of  $G$ .

# THE 13-MODULAR CHARACTER TABLE OF $SL_3(3)$

Let  $G = SL_3(3)$ . Then  $|G| = 5616 = 2^4 \cdot 3^3 \cdot 13$ .

## EXAMPLE (THE 13-MODULAR CHARACTER TABLE OF $SL_3(3)$ )

	1a	2a	3a	3b	4a	6a	8a	8b
$\varphi_1$	1	1	1	1	1	1	1	1
$\varphi_2$	11	3	2	-1	-1	0	-1	-1
$\varphi_3$	13	-3	4	1	1	0	-1	-1
$\varphi_4$	16	0	-2	1	0	0	0	0
$\varphi_5$	26	2	-1	-1	2	-1	0	0
$\varphi_6$	26	-2	-1	-1	0	1	$\sqrt{-2}$	$-\sqrt{-2}$
$\varphi_7$	26	-2	-1	-1	0	1	$-\sqrt{-2}$	$\sqrt{-2}$
$\varphi_8$	39	-1	3	0	-1	-1	1	1

# GOALS AND RESULTS

## AIM

*Describe all Brauer character tables of all finite simple groups and related finite groups.*

In contrast to the case of ordinary character tables (cf. Lecture 3), this is wide open:

- 1 For alternating groups: complete up to  $A_{17}$
- 2 For groups of Lie type: only partial results
- 3 For sporadic groups up to McL and other “small” groups (of order  $\leq 10^9$ ): *An Atlas of Brauer Characters*, Jansen, Lux, Parker, Wilson, 1995

More information is available on the web site of the

**Modular Atlas Project:**

(<http://www.math.rwth-aachen.de/~MOC/>)

# THE DECOMPOSITION NUMBERS

For  $\chi \in \text{Irr}(G) = \{\chi_1, \dots, \chi_m\}$ , write  $\hat{\chi}$  for the restriction of  $\chi$  to  $G_{\ell'}$ .

Then there are integers  $d_{ij} \geq 0$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$  such that  $\hat{\chi}_i = \sum_{j=1}^n d_{ij} \varphi_j$ .

These integers are called the **decomposition numbers** of  $G$  modulo  $\ell$ .

The matrix  $D = [d_{ij}]$  is the **decomposition matrix** of  $G$ .

## PROPERTIES OF BRAUER CHARACTERS

Two **irreducible**  $k$ -representations are equivalent if and only if their Brauer characters are equal.

$\text{IBr}_\ell(G)$  is linearly independent (in  $\text{Maps}(G_{\ell'}, K)$ ) and so the decomposition numbers are uniquely determined.

The elementary divisors of  $D$  are all 1 (i.e., the decomposition map defined by  $\chi \mapsto \hat{\chi}$  is surjective). Thus:

Knowing  $\text{Irr}(G)$  and  $D$  is equivalent to knowing  $\text{Irr}(G)$  and  $\text{IBr}_\ell(G)$ .

If  $G$  is  $\ell$ -soluble,  $\text{Irr}(G)$  and  $\text{IBr}_\ell(G)$  can be sorted such that  $D$  has shape

$$D = \begin{bmatrix} I_n \\ D' \end{bmatrix},$$

where  $I_n$  is the  $(n \times n)$  identity matrix (Fong-Swan theorem).



# UNIPOTENT BRAUER CHARACTERS

The concept of decomposition numbers can be used to define unipotent Brauer characters of a finite reductive group.

Let  $(\mathbf{G}, F)$  be a finite reductive group of characteristic  $p$ .  
(Recall that  $\text{char}(k) = \ell \neq p$ .)

Recall that

$$\text{Irr}^u(\mathbf{G}) = \{\chi \in \text{Irr}(\mathbf{G}) \mid \chi \text{ occurs in } R_{\mathbf{T},1}^{\mathbf{G}} \text{ for some max. torus } \mathbf{T} \text{ of } \mathbf{G}\}.$$

This yields a definition of  $\text{IBr}_{\ell}^u(\mathbf{G})$ .

## DEFINITION (UNIPOTENT BRAUER CHARACTERS)

$$\text{IBr}_{\ell}^u(\mathbf{G}) = \{\varphi_j \in \text{IBr}_{\ell}(\mathbf{G}) \mid d_{ij} \neq 0 \text{ for some } \chi_i \in \text{Irr}^u(\mathbf{G})\}.$$

*The elements of  $\text{IBr}_{\ell}^u(\mathbf{G})$  are called the **unipotent Brauer characters** of  $\mathbf{G}$ .*

A simple  $k\mathbf{G}$ -module is **unipotent**, if its Brauer character is.

# JORDAN DECOMPOSITION OF BRAUER CHARACTERS

The investigations are guided by the following main conjecture.

## CONJECTURE

*Suppose that  $Z(\mathbf{G})$  is connected. Then there is a labelling*

$$\mathrm{IBr}_\ell(\mathbf{G}) \leftrightarrow \{\varphi_{s,\mu} \mid s \in \mathbf{G}^* \text{ semisimple}, \ell \nmid |s|, \mu \in \mathrm{IBr}_\ell^u(C_{\mathbf{G}^*}(s))\},$$

*such that  $\varphi_{s,\mu}(1) = |G^* : C_{G^*}(s)|_p \mu(1)$ .*

*Moreover,  $D$  can be computed from the decomposition numbers of **unipotent** characters of the various  $C_{\mathbf{G}^*}(s)$ .*

Known to be true for  $\mathrm{GL}_n(q)$  (Dipper-James, 1980s) and in many other cases (Bonnafé-Rouquier, 2003).

The truth of this conjecture would reduce the computation of decomposition numbers to unipotent characters.

Consequently, we will restrict to this case in the following.

# THE UNIPOTENT DECOMPOSITION MATRIX

Put  $D^u :=$  restriction of  $D$  to  $\text{Irr}^u(G) \times \text{IBr}_\ell^u(G)$ .

**THEOREM (GECK-H., 1991; GECK, 1993)**

*(Some conditions apply.)*

$|\text{Irr}^u(G)| = |\text{IBr}_\ell^u(G)|$  and  $D^u$  is invertible over  $\mathbb{Z}$ .

**CONJECTURE (GECK, 1997)**

*(Some conditions apply.) With respect to suitable orderings of  $\text{Irr}^u(G)$  and  $\text{IBr}_\ell^u(G)$ ,  $D^u$  has shape*

$$\begin{bmatrix} 1 & & & \\ \star & 1 & & \\ \vdots & \vdots & \ddots & \\ \star & \star & \star & 1 \end{bmatrix}.$$

This would give a canonical bijection  $\text{Irr}^u(G) \longleftrightarrow \text{IBr}_\ell^u(G)$ .

## ABOUT GECK'S CONJECTURE

Geck's conjecture on  $D^u$  is known to hold for

- $GL_n(q)$  (Dipper-James, 1980s)
- $GU_n(q)$  (Geck, 1991)
- $G$  a classical group and  $\ell$  "linear" (Gruber-H., 1997)
- $Sp_4(q)$  (White, 1988 – 1995)
- $Sp_6(q)$  (An-H., 2006)
- $G_2(q)$  (Shamash-H., 1989 – 1992)
- $F_4(q)$  (Köhler, 2006)
- $E_6(q)$  (Geck-H., 1997; Miyachi, 2008)
- Steinberg triality groups  ${}^3D_4(q)$  (Geck, 1991)
- Suzuki groups (for general reasons)
- Ree groups (Himstedt-Huang, 2009)

# LINEAR PRIMES, I

Let  $(\mathbf{G}, F)$  be a finite reductive group, where  $F = F_q$  is the standard Frobenius morphism  $(a_{ij}) \mapsto (a_{ij}^q)$ .

Put  $e := \min\{i \mid \ell \text{ divides } q^i - 1\}$ , the order of  $q$  in  $\mathbb{F}_\ell^*$ .

If  $G$  is classical ( $\neq \mathrm{GL}_n(q)$ ) and  $e$  is odd,  $\ell$  is **linear** for  $G$ .

## EXAMPLE

$G = \mathrm{SO}_{2m+1}(q)$ ,  $|G| = q^{m^2}(q^2 - 1)(q^4 - 1) \cdots (q^{2m} - 1)$ .

If  $\ell \parallel |G|$  and  $\ell \nmid q$ , then  $\ell \mid q^{2^d} - 1$  for some minimal  $d$ .

Thus  $\ell \mid q^d - 1$  ( $\ell$  linear and  $e = d$ ) or  $\ell \mid q^d + 1$  ( $e = 2d$ ).

Now  $\mathrm{Irr}^u(G)$  is a union of Harish-Chandra series  $\mathcal{E}_1, \dots, \mathcal{E}_r$ .

## THEOREM (FONG-SRINIVASAN, 1982, 1989)

Suppose that  $G \neq \mathrm{GL}_n(q)$  is classical and that  $\ell$  is linear.

Then  $D^u = \mathrm{diag}[\Delta_1, \dots, \Delta_r]$  with square matrices  $\Delta_i$  corresponding to  $\mathcal{E}_i$ .



# THE $v$ -SCHUR ALGEBRA

Let  $v$  be an indeterminate and put  $A := \mathbb{Z}[v, v^{-1}]$ .

Dipper and James (1989) have defined a remarkable  $A$ -algebra  $\mathcal{S}_{A,v}(S_n)$ , called the **generic  $v$ -Schur algebra**, such that:

- 1  $\mathcal{S}_{A,v}(S_n)$  is free and of finite rank over  $A$ .
- 2  $\mathcal{S}_{A,v}(S_n)$  is constructed from the generic Iwahori-Hecke algebra  $\mathcal{H}_{A,v}(S_n)$ , which is contained in  $\mathcal{S}_{A,v}(S_n)$  as a subalgebra (with a different unit).
- 3  $\mathbb{Q}(v) \otimes_A \mathcal{S}_{A,v}(S_n)$  is a quotient of the quantum group  $\mathcal{U}_v(\mathfrak{gl}_n)$ .

# THE $q$ -SCHUR ALGEBRA

Let  $G = \mathrm{GL}_n(q)$ .

Then  $D^u = (d_{\lambda, \mu})$ , with  $\lambda, \mu \in \mathcal{P}_n$ .

Let  $\mathfrak{S}_{A, \nu}(\mathcal{S}_n)$  be the  $\nu$ -Schur algebra, and let  $\mathfrak{S} := \mathfrak{S}_{k, q}(\mathcal{S}_n)$  be the finite-dimensional  $k$ -algebra obtained by specializing  $\nu$  to the image of  $q \in k$ .

This is called the  $q$ -Schur algebra, and satisfies:

- 1  $\mathfrak{S}$  has a set of (finite-dimensional) **standard modules**  $\mathbf{S}^\lambda$ , indexed by  $\mathcal{P}_n$ .
- 2 The simple  $\mathfrak{S}$ -modules  $\mathbf{D}^\lambda$  are also labelled by  $\mathcal{P}_n$ .
- 3 If  $[\mathbf{S}^\lambda : \mathbf{D}^\mu]$  denotes the multiplicity of  $\mathbf{D}^\mu$  as a composition factor in  $\mathbf{S}^\lambda$ , then  $[\mathbf{S}^\lambda : \mathbf{D}^\mu] = d_{\lambda, \mu}$ .

As a consequence, the  $d_{\lambda, \mu}$  are bounded independently of  $q$  and of  $\ell$ .

# CONNECTIONS TO DEFINING CHARACTERISTICS, I

Let  $\mathcal{S}_{k,q}(\mathcal{S}_n)$  be the  $q$ -Schur algebra introduced above.

Suppose that  $\ell \mid q - 1$  so that  $q \equiv 1 \pmod{\ell}$ .

Then  $\mathcal{S}_{k,q}(\mathcal{S}_n) \cong \mathcal{S}_k(\mathcal{S}_n)$ , where  $\mathcal{S}_k(\mathcal{S}_n)$  is the **Schur algebra** studied by J. A. Green (1980).

A partition  $\lambda$  of  $n$  may be viewed as a dominant weight of  $\mathrm{GL}_n(k)$  [ $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \leftrightarrow \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \dots + \lambda_m \varepsilon_m$ ].

Thus there are corresponding  $k\mathrm{GL}_n(k)$ -modules  $V(\lambda)$  and  $L(\lambda)$ .

If  $\lambda$  and  $\mu$  are partitions of  $n$ , we have

$$[V(\lambda) : L(\mu)] = [\mathbf{S}^\lambda : \mathbf{D}^\mu] = d_{\lambda,\mu}.$$

The first equality comes from the significance of the Schur algebra, the second from that of the  $q$ -Schur algebra.

## CONNECTIONS TO DEFINING CHARACTERISTICS, II

Thus the  $\ell$ -modular decomposition numbers of  $\mathrm{GL}_n(q)$  for prime powers  $q$  with  $\ell \mid q - 1$ , determine the composition multiplicities of **certain** simple modules  $L(\mu)$  in **certain** Weyl modules  $V(\lambda)$  of  $\mathrm{GL}_n(k)$ , namely if  $\lambda$  and  $\mu$  are partitions of  $n$ .

### FACTS (SCHUR, GREEN)

*Let  $\lambda$  and  $\mu$  be partitions with at most  $n$  parts.*

- 1  $[V(\lambda) : L(\mu)] = 0$ , if  $\lambda$  and  $\mu$  are partitions of different numbers.
- 2 If  $\lambda$  and  $\mu$  are partitions of  $r \geq n$ , then the composition multiplicity  $[V(\lambda) : L(\mu)]$  is the same in  $\mathrm{GL}_n(k)$  and  $\mathrm{GL}_r(k)$ .

Hence the  $\ell$ -modular decomposition numbers of **all**  $\mathrm{GL}_r(q)$ ,  $r \geq 1$ ,  $\ell \mid q - 1$  determine the composition multiplicities of **all** Weyl modules of  $\mathrm{GL}_n(k)$ .

## CONNECTIONS TO SYMMETRIC GROUP REPR'S

As for the Schur algebra, there are standard  $kS_n$ -modules  $S^\lambda$ , called **Specht modules**, labelled by the partitions  $\lambda$  of  $n$ .

The simple  $kS_n$ -modules  $D^\mu$  are labelled by the  **$\ell$ -regular** partitions  $\mu$  of  $n$  (no part of  $\mu$  is repeated  $\ell$  or more times).

The  $\ell$ -modular decomposition numbers of  $S_n$  are the  $[S^\lambda : D^\mu]$ .

### THEOREM (JAMES, 1980)

$[S^\lambda : D^\mu] = [V(\lambda) : L(\mu)]$ , if  $\mu$  is  $\ell$ -regular (notation from  $GL_n(k)$  case).

### THEOREM (ERDMANN, 1995)

For partitions  $\lambda, \mu$  of  $n$ , there are  $\ell$ -regular partition  $t(\lambda), t(\mu)$  of  $\ell n + (\ell - 1)n(n - 1)/2$  such that

$$[V(\lambda) : L(\mu)] = [S^{t(\lambda)} : D^{t(\mu)}].$$

## AMAZING CONCLUSION

Recall that  $\ell$  is a fixed prime and  $k$  an algebraically closed field of characteristic  $\ell$ .

Each of the following three families of numbers can be determined from any one of the others:

- 1 The  $\ell$ -modular decomposition numbers of  $S_n$  for all  $n$ .
- 2 The  $\ell$ -modular decomposition numbers of the unipotent characters of  $GL_n(q)$  for all prime powers  $q$  with  $\ell \mid q - 1$  and all  $n$ .
- 3 The composition multiplicities of the simple  $kGL_n(k)$ -modules in the Weyl modules of  $GL_n(k)$  for all  $n$ .

Thus all these problems are really hard.

# JAMES' CONJECTURE

Let  $G = \mathrm{GL}_n(q)$ . Recall that  $e = \min\{i \mid \ell \text{ divides } q^i - 1\}$ .  
James has computed all matrices  $D^u$  for  $n \leq 10$ .

## CONJECTURE (JAMES, 1990)

*If  $e\ell > n$ , then  $D^u$  only depends on  $e$  (neither on  $\ell$  nor  $q$ ).*

## THEOREM

*(1) The conjecture is true for  $n \leq 10$  (James, 1990).*

*(2) If  $\ell \gg 0$ ,  $D^u$  only depends on  $e$  (Geck, 1992).*

In fact, Geck proved  $D^u = D_e D_\ell$  for two square matrices  $D_e$  and  $D_\ell$ , and that  $D_\ell = I$  for  $\ell \gg 0$ .

## THEOREM (LASCoux-LECLERC-THIBON; ARIKI; VARAGNOLO-VASSEROT (1996 – 99))

*The matrix  $D_e$  can be computed from the canonical basis of a certain highest weight module of the quantum group  $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ .*

# A UNIPOTENT DECOMPOSITION MATRIX FOR $GL_5(q)$

Let  $G = GL_5(q)$ ,  $e = 2$  (i.e.,  $\ell \mid q + 1$  but  $\ell \nmid q - 1$ ), and assume  $\ell > 2$ . Then  $D^u$  equals

---

(5)	1				
(4, 1)		1			
(3, 2)			1		
(3, 1 <sup>2</sup> )	1		1	1	
(2 <sup>2</sup> , 1)			1	1	1
(2, 1 <sup>3</sup> )		1			1
(1 <sup>5</sup> )	1		1		1

---

The triangular shape defines  $\varphi_\lambda, \lambda \in \mathcal{P}_5$ .

# ON THE DEGREE POLYNOMIALS

The degrees of the  $\varphi_\lambda$  are “polynomials in  $q$ ”.

$\lambda$	$\varphi_\lambda(1)$
(5)	1
(4, 1)	$q(q+1)(q^2+1)$
(3, 2)	$q^2(q^4+q^3+q^2+q+1)$
(3, 1 <sup>2</sup> )	$(q^2+1)(q^5-1)$
(2 <sup>2</sup> , 1)	$(q^3-1)(q^5-1)$
(2, 1 <sup>3</sup> )	$q(q+1)(q^2+1)(q^5-1)$
(1 <sup>5</sup> )	$q^2(q^3-1)(q^5-1)$

**THEOREM (BRUNDAN-DIPPER-KLESHCHEV, 2001)**

*The degrees of  $\chi_\lambda(1)$  and of  $\varphi_\lambda(1)$  as polynomials in  $q$  are the same.*

# GENERICITY

Let  $\{G(q) \mid q \text{ a prime power with } \ell \nmid q\}$  be a **series** of finite groups of Lie type, e.g.  $\{GU_n(q)\}$  or  $\{SO_{2m+1}(q)\}$  ( $n$ , respectively  $m$  fixed).

## QUESTION

*Is an analogue of James' conjecture true for  $\{G(q)\}$ ?*





If **yes**, only finitely many matrices  $D^u$  to compute (finitely many  $e$ 's and finitely many "small"  $\ell$ 's).  
The following is a weaker form.

## CONJECTURE

*The entries of  $D^u$  are bounded independently of  $q$  and  $\ell$ .*

This conjecture is known to be true for  $GL_n(q)$  (Dipper-James),  $G$  classical and  $\ell$  linear (Gruber-H., 1997),  $GU_3(q)$ ,  $Sp_4(q)$  (Okuyama-Waki, 1998, 2002).

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