Group Theory and Cryptography

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Overview

Discrete logarithm problem (DLP)

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x	0	1	2	3	4	5	6	7	8	9
g^x	1	2	4	8	5	10	9	7	3	6

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Alice and Bob have the same key: $(g^a)^b = (g^b)^a$.

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- Note: Integers mod n under addition is a bad choice.
- What about non-abelian groups?

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- How do you choose a group G and subgroups A and B?
- Ko et al. suggest using a braid group.

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- Length based attacks work for many instances.
- How can we generate hard instances?
- There are no secure and practical braid based schemes known.

Other ideas

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- Logarithmic signatures are another way to generalise the DLP.
- There is a nice idea due to Anshel, Anshel and Goldfeld.
- There are no group-based cryptosystems that have stood up to long term scrutiny.
- Other *Post-Quantum* cryptosystems might have more potential.
- Let's turn to the cryptanalysis of a specific proposal.

A dollar-firing game

Let $\Gamma = (V, E)$ be a graph. Let $q \in V$ be a vertex.
A configuration: a function $s : V \to \mathbb{Z}$ such that $s(v) \geq 0$ for $v \neq q$, and $\sum_v s(v) = 0$.
A firing: for a fixed $u \in V$, move a dollar along every edge away from u.

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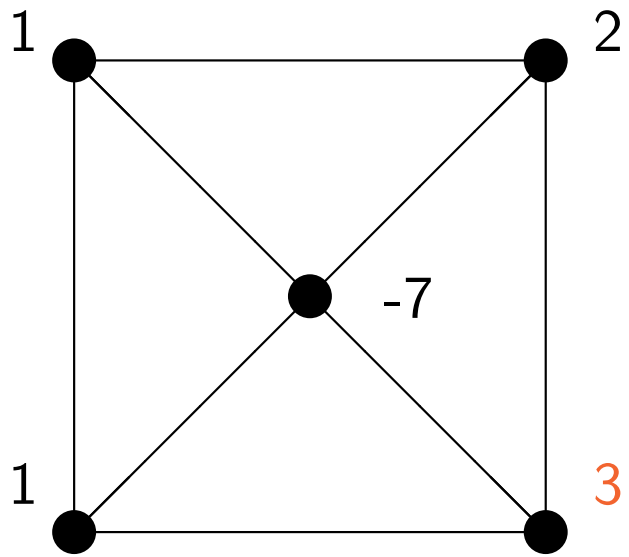

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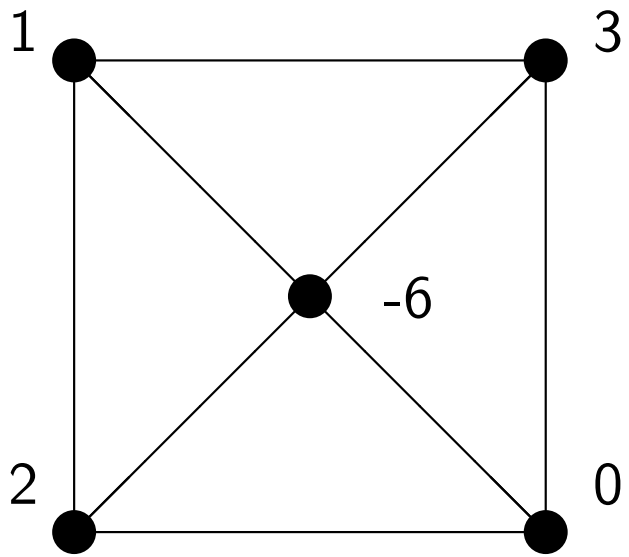

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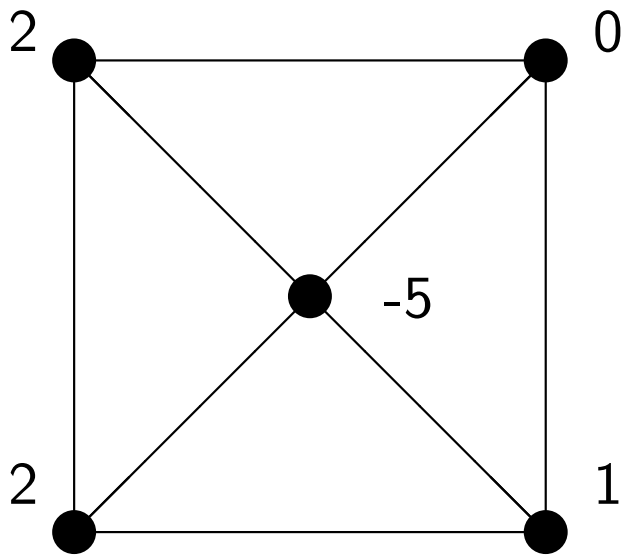

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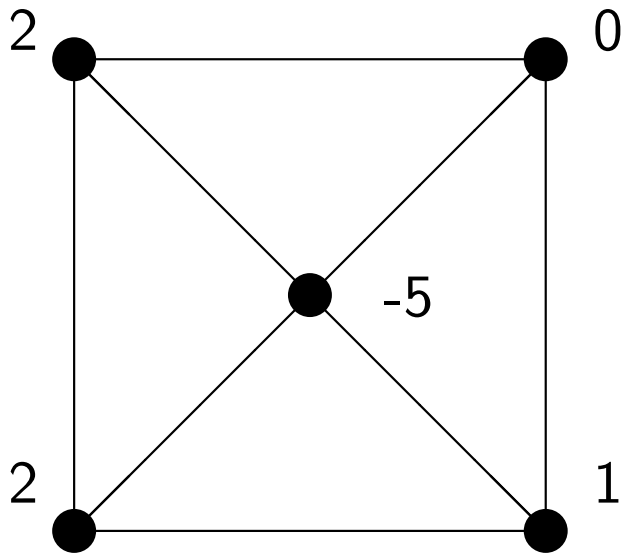

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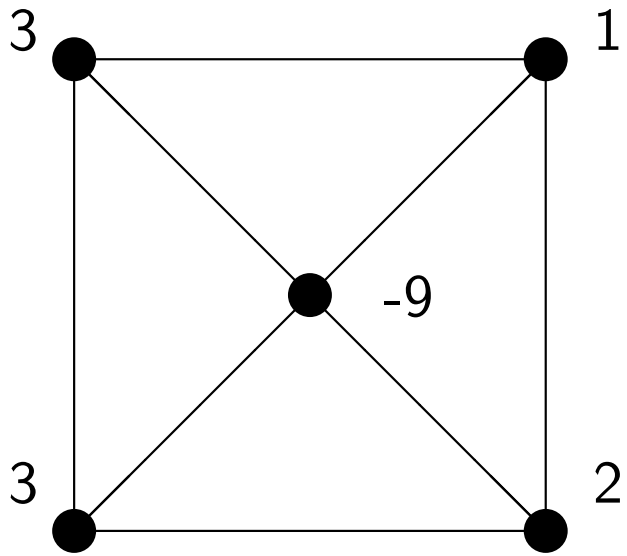

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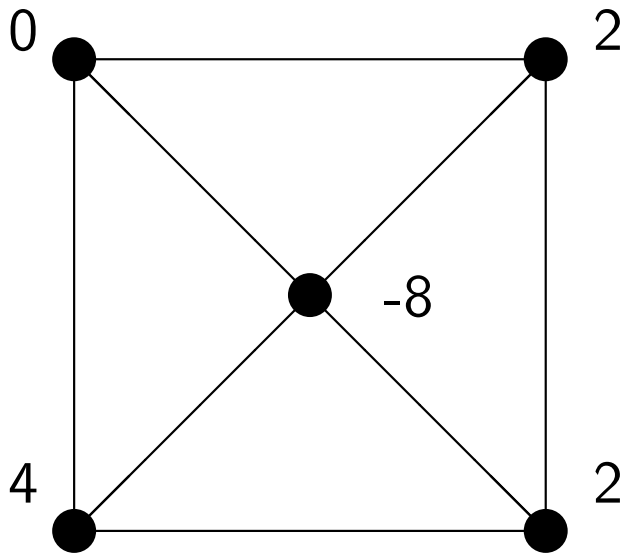

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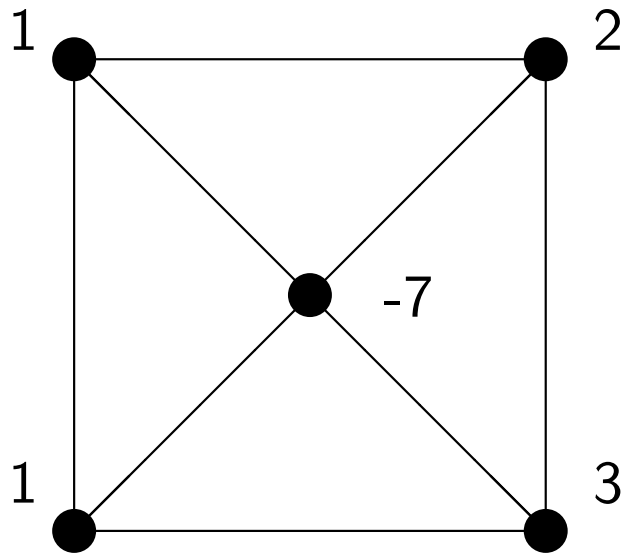

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- The critical group is finite and abelian.
- Addition may be carried out using at most $O(|V|^3)$ firings. [van den Heuvel, *Combin. Probab. Comput.* 2001]

The proposed platform group

Take the wheel graph with vertices $v_1, v_2, \ldots, v_{2n+2}$ on the 'rim', and a 'hub' vertex q.
Remove the spoke at v_{2n+2}, to obtain a graph $W^\dagger$.

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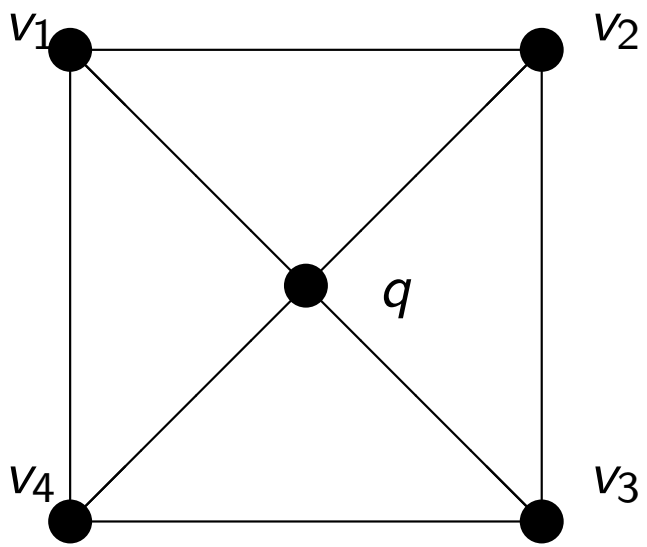

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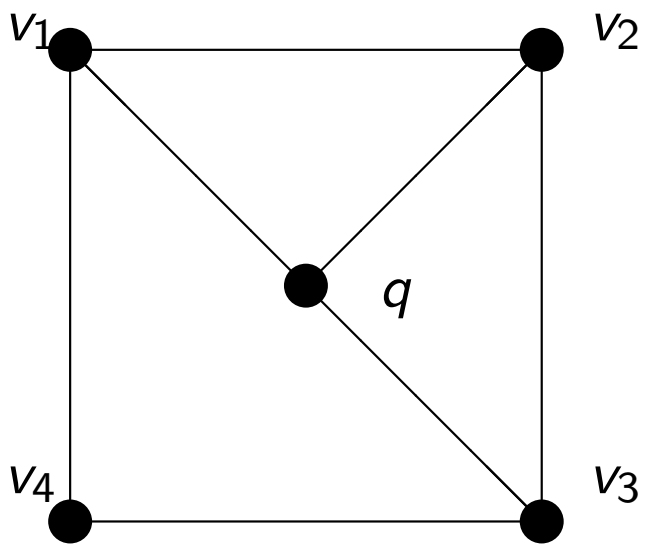

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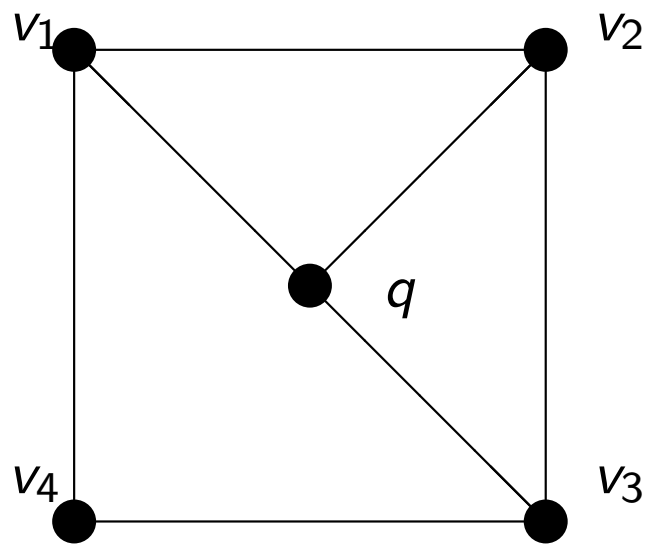

Proposed as a platform group by [Biggs, *Bull. LMS.* 2007]

Advantages of $\mathcal{K}(W^\dagger)$

- Cyclic.
- Concrete representation of elements: $O(n)$ bits.
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- Exponential order: $2\ell_{2n+1}f_{2n+2}$ elements.

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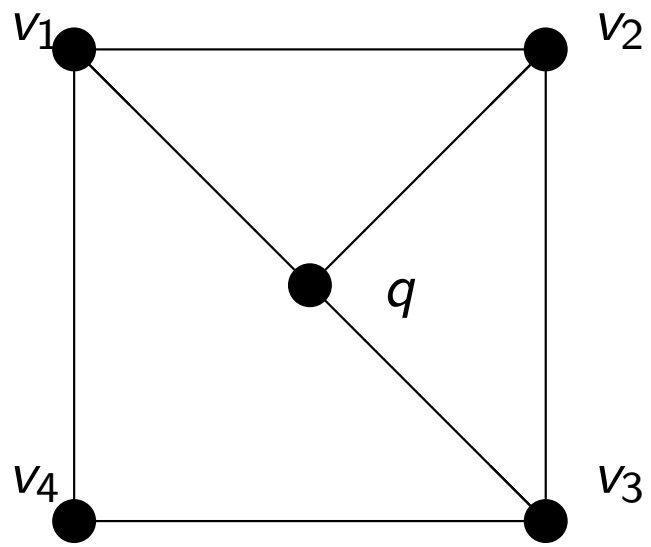
Answer: No.

Another perspective on $\mathcal{K}(W^\dagger)$:



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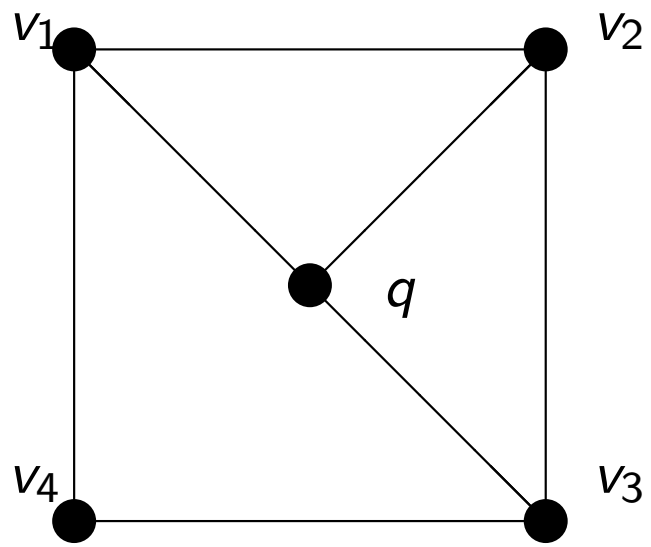

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Firings: $|V|$ relations.

$$3v_1 = v_2 + v_4 \quad 3v_2 = v_1 + v_3$$
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plus the dependent relation $0 = v_1 + v_2 + v_3$.

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Theorem

For any connected graph Γ, $\mathrm{Pic}(\Gamma) \cong \mathcal{K}(\Gamma)$.

(See Biggs, *Bull. LMS*, 1997)

The relations of $\mathrm{Pic}(W^\dagger)$

The relations matrix of the Picard group of $W^\dagger$:

$$Q'' = \begin{pmatrix} 1 & & & & & 1 & -2 \\ -3 & 1 & & & & & 1 \\ 1 & -3 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -3 & 1 & & \\ & & & 1 & -3 & 1 & \\ & & & & 1 & -3 & 1 \end{pmatrix}$$

A cryptanalysis

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- The discrete log problem in G is trivial to solve.

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- The special structure of Q'' can be used to reduce the complexity of the attack to $O(n)$ integer operations.
- The SNF attack applies to any graph, not just $W^{\dagger}$.

Some Links

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S.R. Blackburn, C. Cid, C. Mullan, 'Group theory in cryptography' is available at:

`http://arxiv.org/abs/0906.5545`