On the Regular Semisimple Elements and Primary Classes of $GL(n, q)$

Jamshid Moori$^a$, Ayoub Basheer$^{a,b}$

$^a$ School of Mathematical Sciences, University of KwaZulu Natal, P Bag X01, Scottsville 3209, Pietermaritzburg, South Africa.

$^b$ Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Khartoum, P. O. Box 321, Khartoum, Sudan.

3 Aug 2009
Abstract

In this talk we count the numbers of regular semisimple elements and primary classes of \( GL(n, q) \). The approach used here depends essentially on partitions of positive integers \( \leq n \). We give the numbers of regular semisimple elements and primary classes of \( GL(n, q) \) for \( n \in \{1, 2, \cdots, 6\} \) and see that the number of regular semisimple elements is an integral polynomial in \( q \), while the number of primary classes is a rational polynomial in \( q \).
The Group $GL(n, q)$

- The **General Linear Group** $GL(V)$ is the automorphism group of a vector space $V$.
- If $V$ is a finite $n$–dimensional space defined over a filed $\mathbb{F}$, then $GL(V)$ is identified with $GL(n, \mathbb{F})$.
- We restrict ourselves to the case $\mathbb{F} = \mathbb{F}_q$, the **Galois Field** of $q$ elements, and we denote $GL(n, \mathbb{F}_q)$ by $GL(n, q)$.
- $|GL(n, q)| = \prod_{k=0}^{n-1} (q^n - q^k)$. 

Ayoub Basheer, University of Khartoum  
Groups St Andrews, University of Bath, England
Conjugacy Classes of $GL(n, q)$

Let $f(t) = \sum_{i=0}^{d} a_i t^i \in \mathbb{F}_q[t]$, $a_d = 1$. The $d \times d$ companion matrix $U(f) = U_1(f)$ of $f(t)$ is

$$U_1(f) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -a_1 & -a_2 & \cdots & -a_{d-1}
\end{pmatrix},$$
Conjugacy Classes of $GL(n, q)$

- For any $m \in \mathbb{N}$, let $U_m(f)$ be the $md \times md$ matrix of blocks

$$U_m(f) = \begin{pmatrix}
U_1(f) & I_d & 0 & \cdots & 0 \\
0 & U_1(f) & I_d & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & U_1(f)
\end{pmatrix}.$$

- If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ is a partition of $n$, then $U_{\lambda}(f)$ is defined to be

$$U_{\lambda}(f) = \bigoplus_{i=0}^{k} U_{\lambda_i}(f).$$
Conjugacy Classes of $GL(n, q)$

**Theorem 1 (The Jordan Canonical Form)**

Let $A \in GL(n, q)$ with characteristic polynomial $f_A = f_1^{z_1} f_2^{z_2} \cdots f_k^{z_k}$, where $f_i$, $1 \leq i \leq k$ are distinct irreducible polynomials over $\mathbb{F}_q[t]$ and $z_i$ is the multiplicity of $f_i$. Then $A$ is conjugate to a matrix of the form

$$\bigoplus_{i=1}^{k} U_{\nu_i}(f_i), \text{ where } \nu_i \vdash z_i.$$ 

Thus any conjugacy class of $GL(n, q)$ is parameterized by the data of sequences $(\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$, where for $1 \leq i \leq k$,

$$\sum_{i=1}^{k} z_i d_i = n, \nu_i \vdash z_i, \quad f_i \in \mathbb{F}_q[t] \text{ is irreducible with } \partial f_i = \deg(f_i) = d_i.$$
Conjugacy Classes of $GL(n, q)$

- The integer $k$ is called the *length* of the data.
- Two data $(\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$ and $(\{g_i\}, \{e_i\}, \{w_i\}, \{\mu_i\})$ with lengths $k$ and $k'$ respectively parameterize the same conjugacy class if $k = k'$ and $\exists \sigma \in S_k$ such that
  \[
  w_i = z_{\sigma(i)}, \quad e_i = d_{\sigma(i)}, \quad \mu_i = \nu_{\sigma(i)} \quad \text{and} \quad g_i = f_{\sigma(i)}, \quad \forall i.
  \]
- Two classes of $GL(n, q)$ parameterized by the above data are said to be of the same *type* if $k = k'$ and $\exists \sigma \in S_k$ such that
  \[
  w_i = z_{\sigma(i)}, \quad e_i = d_{\sigma(i)} \quad \text{and} \quad \mu_i = \nu_{\sigma(i)}
  \]
  $(g_i$ and $f_i$ are allowed to differ).
Conjugacy Classes of $GL(n, q)$

Definition 2

Let $c$ be a conjugacy class given by $(\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$ with length $k$, then

1. $c$ is called primary class if and only if $k = 1$.
2. $c$ is called regular class if and only if $l(\nu_i) \leq 1$, $\forall 1 \leq i \leq k$.
3. $c$ is called semisimple class if and only if $l(\nu_i') \leq 1$, $\forall 1 \leq i \leq k$.
4. $c$ is called regular semisimple class if it is both regular and semisimple. Alternatively, a class is regular semisimple if and only if $\nu_i = 1$, $\forall 1 \leq i \leq k$. 

Ayoub Basheer, University of Khartoum

Groups St Andrews, University of Bath, England
Let \( \phi_r(t) = \prod_{i=1}^{r} (1 - t^r) \). For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n \), where each \( \lambda_i \) appears \( m_{\lambda_i} \) times, set \( \phi_\lambda(t) := \prod_{i=1}^{k} \phi_{m_{\lambda_i}}(t) \).

Also if \( \lambda' \) is the conjugate partition of \( \lambda \), let \( n(\lambda) = \sum_{i=1}^{l(\lambda')} \frac{\lambda'_i(\lambda'_i - 1)}{2} \).

Now if \( A \in c = (\{z_i\}, \{d_i\}, \{\nu_i\}, \{f_i\}) \), then by Green [2], we have
Size of Conjugacy Classes of $GL(n, q)$

\[ |C_{GL(n,q)}(A)| = \prod_{i=1}^{k} q^{d_i(z_i+2n(\nu_i))} \phi_{\nu_i}(q^{-d_i}). \]  \hspace{1cm} (1)

It follows that

\[ |C_A| = \frac{\prod_{s=0}^{n-1} (q^n - q^s))}{\prod_{i=1}^{k} q^{d_i(z_i+2n(\nu_i))} \phi_{\nu_i}(q^{-d_i})}. \]  \hspace{1cm} (2)
Number of Regular Semisimple Elements of $GL(n, q)$

Counting the number of the regular semisimple elements of $GL(n, q)$ relies on

- calculating the number of regular semisimple types,
- calculating the number of classes contained in each of the regular semisimple types,
- calculating the number of elements contained in each of the regular semisimple classes.
Number of Regular Semisimple Types

Proposition 3

There is a $1 - 1$ correspondence between the types of classes of regular semisimple elements of $GL(n, q)$ and partitions of $n$.

**Proof.** A regular semisimple class of $GL(n, q)$ must have the form $c = (\{f_i\}, \{d_i\}, \{1\}_k \text{ times}, \{1\}_k \text{ times})$. Thus all regular semisimple classes of the same type define the partition $(d_1, d_2, \cdots, d_k) \vdash n$. Conversely, it is easy to show that any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ defines a type of regular semisimple classes, where a typical class $c$ will have the form $c = (\{f_i\}, \{\lambda_i\}, \{1\}_k \text{ times}, \{1\}_k \text{ times}), 1 \leq i \leq k$. Hence the result. □
Number of Regular Semisimple Classes of $GL(n, q)$

- It turns out that we may denote any type of regular semisimple classes of $GL(n, q)$ by $T^\lambda$ and a typical class by $c^\lambda$ without any ambiguity.

- Consider the other representation of any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ namely $\lambda = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \vdash n$, where $r_i \in \mathbb{N} \cup \{0\}$.

- Recall that by a result of Gauss (see Lidl and Niederreiter [3]), the number of irreducible polynomials of degree $i$ over $\mathbb{F}_q$ is given by

$$l_i(q) = \frac{1}{i} \sum_{d | i} \mu(d) q^{i/d},$$

where $\mu$ is the Möbius function.
Number of Regular Semisimple Classes of $GL(n, q)$

**Proposition 4**

The number of regular semisimple classes of type $\lambda$, which we denote by $F(\lambda)$, is given by

$$F(\lambda) = \left( \prod_{i=1}^{n} \prod_{s=0}^{r_i-1} (l_i(q) - s) \right) / \left( \prod_{i=1}^{n} r_i! \right),$$

where if $r_i - 1 < 0$, then the term $\prod_{s=0}^{r_i-1} (l_i(q) - s)$ is neglected.

**Proof.** See Proposition 5 Moori and Basheer [4].
Number of Regular Semisimple Elements of $GL(n, q)$

**Proposition 5**

Let $c^\lambda$ be a regular semisimple class, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$. Then

$$|c^\lambda| = \left( \prod_{s=0}^{n-1} (q^n - q^s) \right) / \left( \prod_{i=1}^{k} (q^{\lambda_i} - 1) \right).$$

**Proof.** Let $g \in c^\lambda = (\{f_i\}, \{\lambda_i\}, \{1\}_k \text{ times}, \{1\}_k \text{ times}).$ Since $\nu_i = 1$, $\forall 1 \leq i \leq k$, we obtain by substituting in equation (1) that

$$|C_{GL(n,q)}(g)| = \prod_{i=1}^{k} q^{\lambda_i} \phi_1(q^{-\lambda_i}) = \prod_{i=1}^{k} q^{\lambda_i} \left( \frac{q^{\lambda_i} - 1}{q^{\lambda_i}} \right) = \prod_{i=1}^{k} (q^{\lambda_i} - 1).$$

The result follows by equation (2).
Some Corollaries (Moori and Basheer [4])

- For any positive integer $n$, two partitions namely, $\lambda = (1, 1, \ldots, 1) \vdash n$ and $\sigma = (n) \vdash n$ are of particular interest.

- With $q > n$, we have $F(\lambda) = \frac{(q-1)(q-2)\cdots(q-n)}{n!}$ and
  
  $$F(\sigma) = \frac{1}{n} \sum_{d|n} \mu(d) q^{\frac{n}{d}}.$$

- We have $|c^\lambda| = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n-1} \sum_{j=0}^{i} q^j$ and $|c^\sigma| = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n-1} (q^i - 1)$. 
The Main Theorem: Number of Regular Semisimple Elements of $GL(n, q)$

**Theorem 6**

*With* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \equiv 1^{r_1}2^{r_2} \cdots n^{r_n}$ *for* $r_i \in \mathbb{N} \cup \{0\}$, *the number of regular semisimple elements of* $GL(n, q)$ *is given by*

$$
\sum_{\lambda \vdash n} \frac{n-1}{\prod_{s=0}^{n} (q^n - q^s)} \prod_{i=1}^{n} \prod_{s=0}^{r_i-1} (l_i(q) - s) \\
\prod_{i=1}^{k} (q^{\lambda_i} - 1) \prod_{i=1}^{n} r_i!.
$$

**Proof.** Follows from Propositions 3, 4 and 5.
Example

Consider $GL(4, q)$. Corresponds to $(2, 2) = 2^2 \vdash 4$, we have

$$F(2^2) = \left( \prod_{i=1}^{4} \prod_{s=0}^{r_i - 1} (l_i(q) - s) \right) / \left( \prod_{i=1}^{n} r_i! \right) = \frac{q(q^2 - 1)(q - 2)}{8}.$$ 

$$|c^{(2,2)}| = \frac{3}{\prod_{s=0}^{2} (q^4 - q^s)} = q^6(q - 1)(q^2 + 1)(q^3 - 1).$$ 

Hence there are $\frac{q^7(q^4 - 1)(q^3 - 1)(q - 1)(q - 2)}{8}$ regular semisimple elements of type $(2, 2)$. 

Ayoub Basheer, University of Khartoum | Groups St Andrews, University of Bath, England
Example

Repeating the previous work to the other four partitions of 4, we get a total number of regular semisimple elements of $GL(4, q)$ given by

$$q^{16} - 2q^{15} + q^{13} + q^{12} - 2q^{10} - q^9 - q^8 + 2q^7 + q^6.$$ 

For example the group $GL(4, 5)$, which is of order 116,064,000,000 has 9,299,587,000 regular semisimple elements.

In Table 2 of Moori and Basheer [4] we list the number of types, conjugacy classes, elements in each conjugacy class of regular semisimple elements of $GL(n, q)$ for $n = 1, 2, 3, 4, 5, 6$.

The number of regular semisimple elements of $GL(n, q)$ for $n = 1, 2, 3, 4, 5, 6$ is an integral polynomial in $q$. 

Ayoub Basheer, University of Khartoum, Groups St Andrews, University of Bath, England
Number of Primary Classes of $GL(n, q)$

Recall that a class $c = (\{f_i\}, \{d_i\}, \{z_i\}, \{\nu_i\})$ of $GL(n, q)$ with length $k$ is primary if and only if $k = 1$. That is $c = (f, d, \frac{n}{d}, \nu)$ for some $f \in F_{\leq n}$ with degree $d$, $d | n$, and $\nu \vdash \frac{n}{d}$.

**Theorem 7**

The number of primary classes $pc(n, q)$ of $GL(n, q)$ is given by

$$pc(n, q) = \sum_{d|n} |P(\frac{n}{d})| \cdot I_d(q),$$

where $P(j)$ is the set partitions of $j$.

**Proof.** For fixed $d$ and any $\nu \vdash \frac{n}{d}$ we have $I_d(q)$ irreducible polynomials $f$ of degree $d$, that defines a primary class. Hence there are $|P(\frac{n}{d})| \cdot I_d(q)$ classes defined by the fixed integer $d$ and partitions of $\frac{n}{d}$. The result follows by letting $d$ runs over all divisors of $n$. ■

Ayoub Basheer, University of Khartoum
Groups St Andrews, University of Bath, England
Conjugacy Classes of $GL(n, q)$

Regular Semisimple Elements and Primary Classes of $GL(n, q)$

$pc(n, q)$ for $n = 1, 2, \cdots, 6$ and any $q$

Table: Number of primary classes of $GL(n, q)$, $n = 1, 2, 3, 4, 5, 6$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$pc(n, q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(q - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$(q^2 + 3q - 4)/2$</td>
</tr>
<tr>
<td>3</td>
<td>$(q^3 + 8q - 9)/3$</td>
</tr>
<tr>
<td>4</td>
<td>$(q^4 + 3q^2 + 16q - 20)/4$</td>
</tr>
<tr>
<td>5</td>
<td>$(q^5 + 34q - 35)/5$</td>
</tr>
<tr>
<td>6</td>
<td>$(q^6 + 3q^3 + 8q^2 + 54q - 66)/6$</td>
</tr>
</tbody>
</table>

Ayoub Basheer, University of Khartoum
Groups St Andrews, University of Bath, England
Some Corollaries (Moori and Basheer [4])

- There are exactly $l_n(q) = \frac{1}{n} \sum_{d|n} \mu(d) q^{\frac{n}{d}}$ primary regular semisimple classes of $\text{GL}(n, q)$.

- If $n = p'$ is a prime integer (whether $p' = p$, the characteristic of $\mathbb{F}_q$ or not), then there are $l_{p'}(q) = \frac{q^{p'} - q}{p'}$ primary regular semisimple classes of $\text{GL}(p', q)$.

- We have $\left[q^{\frac{p'^2 - p' + 2}{2}} (q^{p'} - 1)^2 \prod_{i=1}^{p'-2} (q^i - 1)\right] / p'$ primary regular semisimple elements of $\text{GL}(p', q)$.

- The group $\text{GL}(p', q)$ has exactly $(q^{p'} + (p' | P(p')| - 1)q - p' | P(p')|) / p'$ primary conjugacy classes.
Abstract
Conjugacy Classes of $GL(n, q)$

Regular Semisimple Elements and Primary Classes of $GL(n, q)$

The Bibliography


Ayoub Basheer, University of Khartoum
Groups St Andrews, University of Bath, England
Acknowledgement

My special thanks and regards addressed to

- my supervisor Professor Jamshid Moori.
- National Research Foundation (NRF) (Prof. Moori’s research grant) and to the African Institute for Mathematical Sciences (AIMS) for the grant holder bursaries.
- Administration of the University of Khartoum (UofK), in particular the Faculty of Mathematical Sciences and to the Principal of UofK, Dr Mohsin H. A. Hashim, Khartoum, Sudan.
- The University of KwaZulu Natal.